ASTEROIDAL SETS AND DOMINATING TARGETS IN GRAPHS

by

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A DISSERTATION

Presented to the Faculty of

The Graduate College at the University of Nebraska

In Partial Fulfilment of Requirements

For the Degree of Doctor of Philosophy

Major: Computer Science

Under the Supervision of Professor Andrew J. Radcliffe

Lincoln, Nebraska

May, 2024

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University of Nebraska, 2024

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The focus of this Ph.D. thesis is on various distance and domination properties in graphs. In particular, we prove strong results about the interactions between asteroidal sets and dominating targets. Our results add to or extend a plethora of results on these properties within the literature. We define the class of strict dominating pair graphs and show structural and algorithmic properties of this class. Notably, we prove that such graphs have diameter 3, 4, or contain an asteroidal quadruple. Then, we design an algorithm to efficiently recognize chordal hereditary dominating pair graphs. We provide new results that describe the internal structure of these graphs, and prove that asteroidal quadruples may provide diameter bounds. Then, we extend the notion of polarity to dominating targets by defining the concept of polar targets. We investigate dominating targets in cycle graphs and show that they cannot have polar targets. Then, we provide a sufficient condition for a graph to have a polar target of size 3.

DEDICATION

To my wife, Haifaa, for her years of patience and love, as well as our children, Aria and Mikaeel.

ACKNOWLEDGMENTS

Some results presented in this thesis are published or are pending publication. Several results herein are presented for the first time, and will make their way to future publications. I would like to take the opportunity to thank my advisor, Dr. Andrew J. Radcliffe, for his invaluable support. This thesis would not be possible without our collaboration.

Table of Contents

List of Figures

Chapter 1

Introduction

Distance and domination are amongst the most basic concepts in mathematics. Whereas these terms are usually understood in their geometric interpretations, they take on different meanings in combinatorical structures with a non-euclidean nature. For example, metric distances are irrelevant in many data structures used in computer science because of the inherent arrangement of memory addresses.

Most applications require special data structures to efficiently process data or achieve an solution. A *graph* is a ubiquitous data structure in computer science. Graphs allow computer scientists to model systems consisting of objects and relations between those objects, and thus efficient algorithms are required to process these graphs. In a graph, those objects are known as *vertices* and their relationships are called *edges*. In this thesis, we assume the graphs are *simple*. In a simple graph, a pair of vertices has at most one edge between them. Therefore, simple graphs can be used to model binary relations. However, many problems are either untractable or it is unknown whether a polynomial-time algorithm exists. Classically NP-complete problems such as the INDEPENDENT SET, HAMILTONICITY and FEEDBACK VERTEX SET are amongst many such problems. It is most likely that no polynomial-time solution exists for these problems unless P = NP, but such a prospect is beyond the scope of this thesis.

There are ways of coping with the intractability of the aforementioned problems. One method is to utilize approximation algorithms that are guaranteed to offer a solution quickly within a known margin of error. Another method is to "restrict" the general graph to that of some special graph class. In this thesis, the focus is strongly placed on the second method.

1.1 Recognition Algorithms

In order to solve graph theoretical problems, we investigate structural properties of special graph class families. An important problem related to special graph classes is that of RECOGNITION, in which one determines whether a given graph belongs to a particular graph class.

We define several terms that will be used throughout this thesis. By G we denote a simple, undirected, and finite graph with n vertices. As is standard, we let V denote the vertex set of G and E denote the edge set of G. Let $S \subseteq V$ be a subset of vertices of a graph $G = (V, E)$. The *induced subgraph* $G[S]$ is the graph whose vertex set is S and whose edge set consists of all of the edges in E that have both endpoints in S.

We define the notion of a *forbidden* subgraph with respect to a graph class by borrowing the following definition from a survey by Brandstädt et al. [5].

Definition 1. *Let $\mathcal{F}$ be a family of graphs. A graph G is $\mathcal{F}$-free (i.e. F is forbidden in G) if G contains no induced subgraph isomorphic to a member of $\mathcal{F}$.*

The path graph formed by k vertices and $k-1$ edges is denoted by P_k. Consider a graph class that is P_k-free. In order to recognize whether a graph is P_k-free, it is sufficient to iterate through every set $S \subseteq G$ of k vertices in V and compare edge incidence in order to determine if S is an induced P_k. If no such set S exists, then

G is P_k-free. Certainly, we would call any such methodology a "brute-force" whose time-complexity for recognition is lower-bound by the size of S.

Any suboperation that requires brute-force in an algorithm can be considered a bottle-neck to the efficiently of the algorithm. This is especially true in graph classes that are characterized by forbidden subgraphs, but is also true with respect to the computing various properties of graphs. Most broadly, in this thesis we will exploit the extremal and structural properties of certain graph classes in order to avoid utilizing any brute-force approach. Many of our lemmas in Chapter 2 build up to a recognition algorithm.

1.2 Distance and Domination

Applications to distance and domination are legion; for example, the longest of all distances in a graph is its diameter, and the diameter has an immediate application in communication networks where the furthest distance between hosts and clients represents the longest communications distance and slowest time to transfer data between any two destination. Moreover, a small dominating set in a communication network can be considered a resilient "backbone". This dominating set can be reinforced in a cost-effective manner in order to ensure that every server can be reached despite adversarial conditions (i.e. servers may be prone to fail). Many of our domination-related results are motivated by graphs as models for communication networks.

In this thesis, most graphs given have nice distance or domination properties. We say that a vertex u is adjacent to a vertex v if there exists an edge connecting u to v in E. We give a much more strict property of a set in G.

Definition 2. *Given a graph $G = (V, E)$, a subgraph $S \subseteq G$ dominates a set $A \subseteq G$*

if every vertex in A is contained in S or is adjacent to some vertex in S.

We consider two graph structures that capture certain useful notions of distance and domination: the *dominating targets* and *asteroidal sets*. We are especially interested in graphs in which the sizes of these sets is bounded. These graphs have an interesting and rich structure that can be exploited to compute certain properties of graphs more efficiently.

A dominating target is a set of vertices $S \subseteq G$ such that every connected superset of S is a dominating set of G. Therefore, the identification of a dominating target T guarantees that the removal of vertices from G provides a graph having a dominating superset of T so long as such a removal does not disconnect T. Therefore, we might say that a small dominating target is a "domination-preserving" characteristic of a graph.

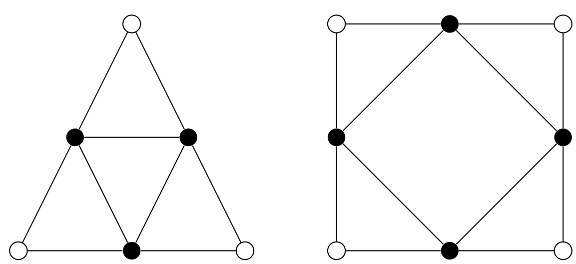

Figure 1.1: Graphs having asteroidal sets of size 3 and 4, respectively, as shown by the unfilled vertices.

An *asteroidal triple* is a triple of vertices such that the removal of the neighborhood of one vertex does not disconnect the other two. Every pair of an asteroidal triple has a path between them that avoids dominating the third. Thus, in a sense, asteroidal triples are inherently domination "avoiding" as they provide paths that can be traversed to avoid the domination of a particular vertex. We also study a generalization of asteroidal triples known as asteroidal sets, but define them slightly differently in Chapters 2 and 3 (in the former, we do not consider asteroidal sets with size 2 or less). Fig. 1.1 provides two examples of asteroidal sets in graphs. Moreover, asteroidal quadruples (asteroidal sets of size 4) are a natural structural consequence of certain distance properties. For example, Gutierrez [22] showed that they exist in directed path graphs that are non-rooted path graphs. We will come to a similar result in Chapter 2.

1.3 Special Graph Classes

In this section, we will provide definitions of several famous and well-studied graph classes. These graph classes provide special edge cases to most of our results because we mostly study their various superclasses and counterparts. We complete this section by foreshadowing our main results.

Definition 3. *A graph is an* interval *graph if it has an intersection model consisting of intervals on a straight line.*

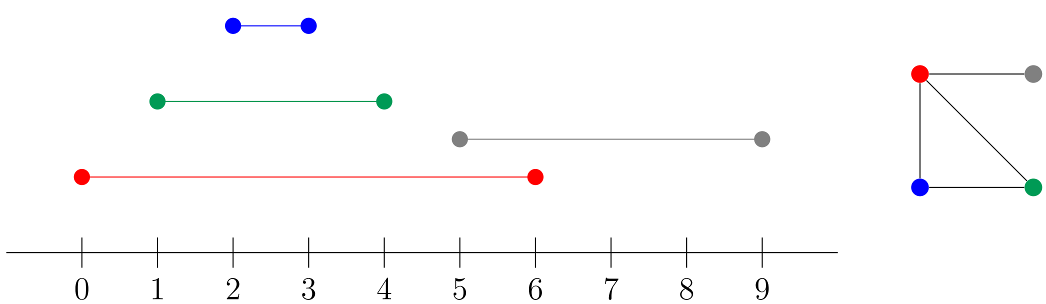

Figure 1.2: An intersection model of four intervals on a straight line and the graph with four vertices that it denotes.

Interval graphs have played a major role in scheduling problems and have were independently discovered within several research disciplines, including bioinformatics and geology [14]. The fact that they have a dominating shortest path [8] has been generalized and applied in a variety of fields, including sequence similarity networks of reads corresponding to singular regions of genomes [26].

One particularly important property of interval graphs is that they are asteroidal triple-free (AT-free). The next graph class is not necessarily AT-free but is vital nonetheless.

Definition 4. *A graph is* chordal *if every cycle of length at least 4 has a chord.*

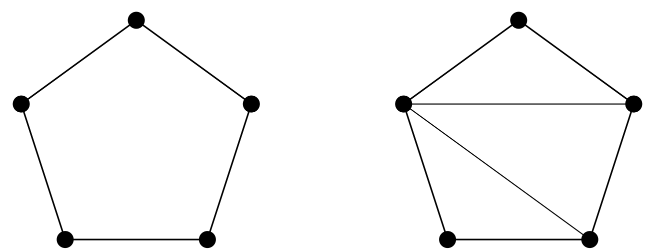

Figure 1.3: The left graph is a non-chordal whereas the right graph is chordal.

The above graph classes are in some ways more restricted than the graph classes we will be studying (see Fig 1.4). Regardless, many algorithmic problems have polynomial-time solutions in AT-free graphs and graphs with bounded sizes on their asteroidal sets.

1.3.1 Known Results and Applications

We will survey some of the known literature: Hempel et al. [15] showed that HAMILTONICITY is polynomial if the AT-free graph is claw-free, although the problem remains open in AT-free graphs in general. AT-free graphs also have dominating targets of size 2 or less [16]. Stacho [25] proved that these graphs can be 3-colored in polynomial-time. Notably, Alcón studied asteroidal sets specifically in chordal graphs [3] and Golovach et al. [13] discovered a polynomial-time algorithm to compute s-clubs (sets of vertices in the graph that induce diameter at most s) in AT-free graphs.

Brandstädt et al. [4] strengthened the connection between AT-free graphs and ad hoc networks by studying r-coloring complexity as a model for frequency assignment in multihop radio networks. Moreover, tree spanners and routing schemes were studied in AT-free graphs in order to more generally model wireless and sensor networks by Yan [27].

This thesis is organized in the following way. In Chapter 2, we study two generalizations of the asteroidal-free graphs: *hereditary dominating pair* and *diametral path*. We show that chordal $\cap$ hereditary dominating pair graphs can be efficiently recognized without using a brute-force operation. Many of our structural results provide necessary conditions for the existence of asteroidal quadruples.

The notion of *polarity* generally refers to the partitioning of subsets of vertices in a graph to satisfy some property. For example, the *split graphs* (graphs whose vertices can be partitioned into an independent set and a clique and are precisely the chordal $\cap$ co-chordal graphs) have also been independently coined *polar graphs* in the literature [11]. In Chapter 3, we investigate notions of polarity in graphs with given dominating targets and asteroidal sets. We show that a graph with a triple exhibiting a particular combination of properties causes the graph to exhibit greater stability in the form of so-called *polar targets*. Along the way, we show that asteroidal sets provide strong restrictions on the connected supersets of a given dominating target. We show that the class of graphs having an asteroidal set whose sizes exceeds that of any dominating target are somewhat rare, potentially a cause for a vulnerability in a network, but mathematically interesting.

Chapters 2 and 3 have been reproduced from published [2] or submitted articles and stand on their own as complete manuscripts apart from this thesis.

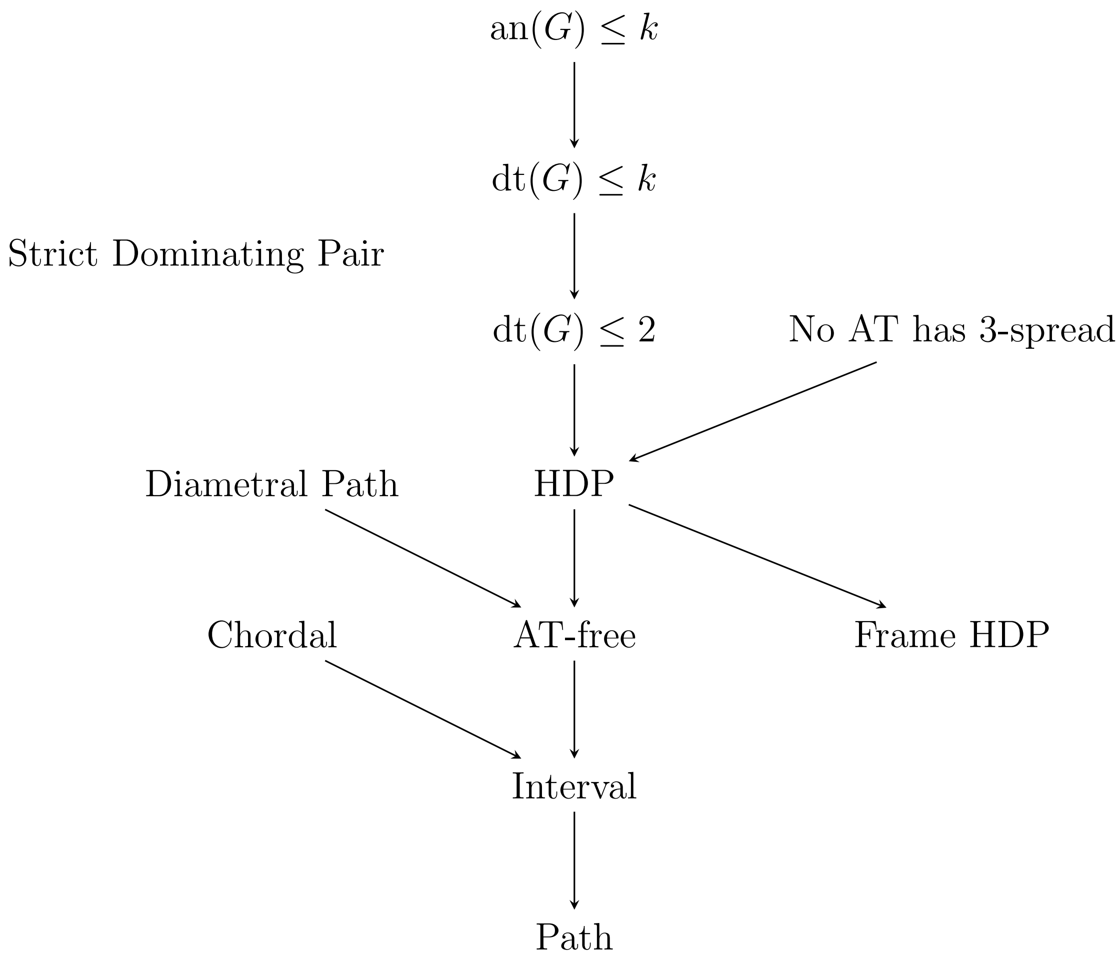

Figure 1.4: A diagram that depicts the inclusion relationship of the graph classes studied in this thesis. For any members $\mathcal{G} \to \mathcal{H}$, we have that $\mathcal{G}$ is a superclass of $\mathcal{H}$.

Chapter 2

Asteroidal Sets and Dominating Paths

An independent set of three vertices is called an asteroidal triple (AT) if there exists a path between any two of them that avoids the neighborhood of the third. Asteroidal triple-free (AT-free) graphs are very well-studied, but some of their various superclasses are not. We study two of these superclasses: hereditary dominating pair (HDP) graphs and diametral path graphs. We correct a mistake that has appeared in the literature claiming that the class of diametral path graphs are a superclass of HDP. More specifically, we show that a graph with a dominating shortest path does not necessarily contain a dominating diametral path. We say a graph is a strict dominating pair graph if it contains a dominating pair but has no dominating diametral path, and we show structural and algorithmic properties of these graphs. To study properties of HDP graphs, we introduce the notion of spread in asteroidal triples. Given a dominating pair, we show that all paths between this pair meet the common neighborhood of some pair from each asteroidal triple. We use these results to improve the best known time complexity for the recognition of chordal HDP graphs.

2.1 Introduction

Asteroidal triple-free (AT-free) graphs capture a common property which imposes the linearity we see in a multitude of classic graph classes. For example, asteroidal triples are forbidden in interval, permutation, and trapezoidal graphs. A famous result by Lekkerkerker and Boland [21] states that interval graphs are exactly the class of chordal AT-free graphs. Several problems that are NP-Complete in general have been shown to have polynomial-time solutions on AT-free graphs, including INDEPENDENT SET and FEEDBACK VERTEX SET [6, 19]. The recognition of AT-free graphs is known to be polynomial [20]. Corneil, Olariu and Stewart [8] showed that AT-free graphs have two key properties: First, all AT-free graphs have a *dominating pair*, a pair of two vertices such that any path between them dominates the vertex set of the graph. Second, all AT-free graphs have a *dominating diametral path*. Because the class of AT-free graphs is hereditary, these two properties hold true for all connected, induced subgraphs of any AT-free graph. In this thesis, we consider these two properties generalized to their own graph classes, which are not necessarily AT-free.

The class of dominating pair graphs is the largest graph class for which the dominating pair property is hereditary. Dominating pair graphs that have asteroidal triples were investigated by Pržulj, Corneil, and Köhler [24], where the name *hereditary dominating pair* (HDP) was introduced. We will use the term HDP when referring to dominating pair graphs in order to better distinguish them from the weaker class of graphs that *have* a dominating pair but do not necessarily have a dominating pair in every connected, induced subgraph. Polynomial-time algorithms for STEINER SETS, MINIMUM CONNECTED DOMINATING SETS exist for HDP graphs [1]. DOMINATING SET and TOTAL DOMINATING SET have also been solved in polynomial-time [18]. In a general sense, HDP graphs may find an intuitive application as a topology for wireless, ad hoc networks or critical systems because communication may be less susceptible to disruption.

A graph is *diametral path* if it contains a dominating diametral path in every connected, induced subgraph. Recently, RAINBOW VERTEX COLORING was studied on the class of diametral path graphs, drawing parallels to encryption and data security [10]. Although it has been claimed that a graph with a dominating shortest path also has a dominating diametral path [23] and this has been referred to in other literature [9], we show a counterexample. If a graph contains a dominating pair but has no dominating diametral path, then the graph is *strict dominating pair*.

In Sect. 2.3, we analyze strict dominating pair graphs. We prove that the diameter of these graphs is close to the distance of a dominating pair, a result that can be used to quickly compute the diameter if given a dominating pair. We give a necessary condition (Theorem 16) to be in strict dominating pair. We show that strict dominating pair graphs have diameter 3, 4, or have an asteroidal number of at least 4.

In Sect. 2.4, we introduce the notion of spread in asteroidal triples. We show that HDP graphs may not contain asteroidal triples with 3-spread. Intuitively speaking, an asteroidal triple without 3-spread has a pair of vertices that remain "close" in every connected, induced subgraph that contains the given asteroidal triple. So, 3-spread describes a hereditary structure that can be exploited to design algorithms, such as identifying cut sets or points of weakness of a network. This allows us to improve the run-time complexity for the recognition of chordal HDP graphs.

In Sect. 2.5, we study the internal structure of graphs that do not have an asteroidal triple with 3-spread. We prove that asteroidal quadruples may provide diameter bounds in HDP graphs. Then, we investigate eccentricities of dominating pair vertices and provide a sufficient condition for one to have eccentricity equal to the diameter of the graph.

2.2 Preliminaries

Our graph theory notation basically follows that of Golumbic [14]. By G we denote a simple, undirected, and finite graph with n vertices. The vertex set of G is denoted by V and the edge set of G is denoted by E. For any $x \in G$ let $N(x) = \{y : \{x, y\} \in E\}$ be the (open) neighborhood of x and $N[x] = N(x) \cup \{x\}$ the closed neighborhood of x.

A set $S \subseteq V$ *dominates* a set $T \subseteq S$ if every vertex in T is contained in the closed neighborhood of S. We write $u \sim v$ to indicate that vertices u and v are adjacent. More generally, we write $u \sim S$ if there exists $v \in S$ such that $u \sim v$. We denote by $C_N(S)$ the *common neighborhood* of S. i.e.

$$C_N(S) = \bigcap_{v \in S} N(v).$$

A sequence of vertices $P = \langle u = x_0, x_1, \ldots, x_{k-1}, x_k = v\rangle$ is called a *walk* if $x_i \sim x_{i+1}$ for all $i \in \{0, 1, \ldots, k-1\}$. We say P is a path if the vertices $x_0, x_1, \ldots, x_k$ are all distinct. A walk with endpoints u and v may be called a u, v-walk. If $v \sim v'$ then we denote by $P{-}v'$ the walk $\langle u = x_0, x_1, \ldots, x_{k-1}, x_k = v, v'\rangle$. Similarly, if $P = \langle x_0, x_1, \ldots, x_k\rangle$ and $P' = \langle y_0, y_1, \ldots, y_q\rangle$ are walks and $x_k \sim y_0$, then we let $P{-}P' = \langle x_0, \ldots, x_k,\ y_0, \ldots, y_q\rangle$. We write $P[x_i, x_j]$ where $(0 \leq i \leq j \leq k)$ for the subwalk $\langle x_i, \ldots, x_j\rangle$ of P. It is well known that every walk contains a path. In other words, we can extract a u, v-path from the vertex set of every u, v-walk. The *length* of a path is the number of edges it contains.

We say a walk P *meets* some set S if $P \cap S \neq \emptyset$ and that P *avoids* S if $P \cap S = \emptyset$.

The notation $d_G(u,v)$ denotes *geodesic distance*, the length of a shortest path, between the vertices u and v. The *eccentricity* $\mathrm{ecc}(v)$ of a vertex v is the greatest geodesic distance between v and some other vertex in G. The *diameter* of G, denoted also as $\mathrm{diam}(G)$, is the greatest geodesic distance between any two vertices in G. Formally, $\mathrm{diam}(G) = \max\{d_G(u,v) : u,v \in G\}$. A *diametral pair* is a pair of vertices (u,v) such that $d_G(u,v) = \mathrm{diam}(G)$. A *diametral path* is any path whose endpoints form a diametral pair. We use the following generalization of asteroidal triples:

Definition 5. *An* asteroidal set *S is an independent set of at least three vertices such that, for every vertex $v \in S$, there exists a path between any two remaining vertices of $S \setminus \{v\}$ that avoids $N[v]$. We call such a path an* asteroidal path*. The cardinality of the largest asteroidal set in a graph is known as the* asteroidal number *of that graph, denoted by* $\mathrm{an}(G)$*. An* asteroidal triple *is an asteroidal set of size 3. An* asteroidal quadruple *is an asteroidal set of size 4.*

Remark. In the literature, asteroidal sets have been defined for 1 or 2 vertices as well. In this section, we do not consider asteroidal sets of these sizes.

Definition 6. *Two vertices a and b form a* dominating pair (a,b) *if every a,b-path dominates G. A* dominating pair path *is any path between a dominating pair of vertices. A graph is called a* hereditary dominating pair graph *(HDP) if every connected, induced subgraph has a dominating pair.*

It is simple to see that a graph with a dominating pair contains a dominating shortest path. We introduce the notion of *spread* as a characteristic of certain asteroidal triples.

Definition 7. *An asteroidal triple has* k-spread *if between any pair of vertices in the triple there is an induced path of length at least* k *that avoids the neighborhood of the third.*

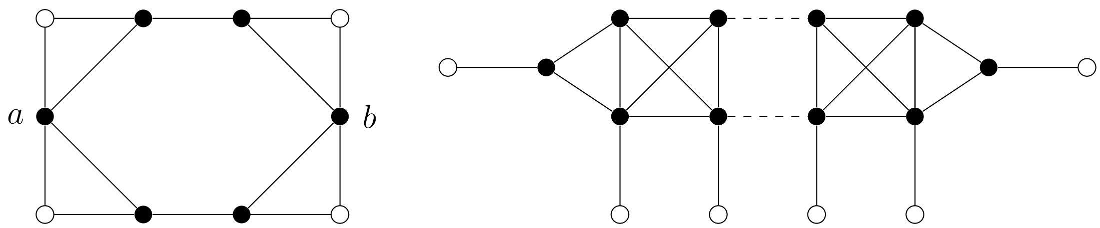

Figure 2.1: On the left is a strict dominating pair graph where (a, b) is a dominating pair and the white vertices form an asteroidal set of size 4. On the right is a diametral path graph that does not contain a dominating pair.

2.3 Dominating Pairs and Diametral Paths

In this section, we study the structural discrepancy between graphs with dominating pairs and graphs with dominating diametral paths. We state new results on the relationship between the two properties and expressly state conditions for when a graph is strict dominating pair. We begin with a correction to the assumption in past literature that any graph with a dominating shortest path contains a dominating diametral path.

Proposition 8. *A graph with a dominating shortest path does not necessarily contain a dominating diametral path.*

Proof. See the left graph in Fig. 2.1 for an example. The pair (a, b) is a dominating pair, and thus every shortest a, b-path is dominating. Notice that (a, b) is not diametral. It is easy to verify that no dominating diametral path exists. □

With respect to the HDP and diametral path graph classes, both are superclasses of AT-free graphs, yet by Proposition 8 neither is a superclass of the other. To prepare for our main results, we present several results that relate the diameter of G to the distance between any dominating pair.

Lemma 9. *Let (a, b) be a dominating pair. There exists a diametral pair with one vertex contained in $N[a]$ and the other contained in $N[b]$.*

Proof. Let P be a shortest a, b-path of the form $\langle a = x_0, x_1, \ldots, x_k = b\rangle$, i.e. $d_G(a, b) = k$. Pair (a, b) is not diametral, otherwise we are done. Let (d_1, d_2) be a diametral pair. Certainly d_1, d_2 are adjacent to P because P is a dominating path by definition of dominating pair (a, b). If $d_1 \sim \langle x_1, \ldots, x_{k-1}\rangle$ and $d_2 \sim \langle x_1, \ldots, x_{k-1}\rangle$ then $d_G(d_1, d_2) \leq k$, a contradiction. Therefore, at least one of d_1 and d_2 is adjacent to a or b. If $d_1 \sim a$ and $d_2 \sim b$ then we are done; therefore, w.l.o.g. suppose $d_1 \sim x_1$ and $d_2 \sim b$. If $d_G(d_1, d_2) = d_G(a, d_2)$ then (a, d_2) is diametral s.t. $d_2 \in N(b)$, and we are done. Notice that $d_G(a, d_2) = k + 1$. So, there exists an a, d_2-path Q of length k. Let $Q = \langle a = u_0, u_1, \ldots, u_k = d_2\rangle$. The path $Q - b$ must be dominating and $d_1 \not\sim b$, so $d_1 \sim Q$. We let $d_1 \sim u_i$ where $i > 0$. Now $d_G(d_1, d_2) \leq 1 + (k - i) < k$, a contradiction to the diameter of G. Thus, $d_1 \in N[a]$ and $d_2 \in N[b]$. □

Immediately, we see the following:

Corollary 10. *Let G contain a dominating pair (a, b). The diameter of G is bounded by the following inequality:*

$$d_G(a, b) \leq \operatorname{diam}(G) \leq d_G(a, b) + 2.$$

Moreover, if G is strict dominating pair then $\operatorname{diam}(G) = d_G(a, b) + 1$ *and there exists $d_1 \in N(a)$ and $d_2 \in N(b)$ s.t. (d_1, d_2) is a diametral pair.*

Proof. By Lemma 9, there exists a diametral pair (d_1, d_2) s.t. $d_1 \in N[a]$ and $d_2 \in N[b]$. If $\operatorname{diam}(G) = d_G(a, b)$ then any a, b-path is a dominating diametral path, a contradiction. If $\operatorname{diam}(G) = d_G(a, b) + 2$, let P be any shortest a, b-path. We must have $d_1 \neq a$ and $d_2 \neq b$. The path $d_1 - P - d_2$ is a diametral path that dominates G because it contains P as a subpath.

In the remaining case, $\operatorname{diam}(G) = d_G(a, b) + 1$. If $d_1 = a$ or $d_2 = b$ then either $P - b$ or $a - P$ is a dominating diametral path, a contradiction. □

Theorem 16 gives an interesting necessary condition, but not a characterization, of strict dominating pair graphs. To simplify the proofs, we introduce the notion of *corner vertices*, vertices that bear witness to the fact that a diametral path is not dominating but a given pair (a, b) is a dominating pair.

Definition 11. *If Q is a non-dominating diametral path in a graph G and (a, b) is a dominating pair, then vertex c is a corner w.r.t. (Q, a) if c is not dominated by Q, an endpoint of Q is adjacent to a, and $c \in N(a)$.*

Note that corners are adjacent to elements of the dominating pair, rather than to intermediate vertices on dominating pair paths. Next, we prove that corner vertices are inevitable in strict dominating pair graphs.

Lemma 12. *Let G have a dominating pair (a, b) and let $k = d_G(a, b)$, but no dominating diametral path (and thus in particular $\operatorname{diam}(G) = k + 1$). Let (d_1, d_2) be a diametral pair s.t. $d_1 \in N(a)$ and $d_2 \in N(b)$. There exists a diametral d_1, d_2-path Q that contains b and a corner vertex c w.r.t. (Q, a).*

Proof. By Corollary 10 we have $d_G(d_1, d_2) = k + 1$. We claim that $d_G(d_1, b) = k$. By Corollary 10, $d_G(d_1, b) \leq k + 1$. If $d_G(d_1, b) < k$ then $d_G(d_1, d_2) < k + 1$ because $d_2 \sim b$, a contradiction. Thus, $d_G(d_1, b) = k$. Similarly, $d_G(a, d_2) = k$. Let M be a shortest d_1, b-path. By assumption, diametral path $M{-}d_2$ is not dominating while dominating pair path $a{-}M$ is dominating. Since $(a{-}M) \setminus (M{-}d_2) = \{a\}$, there exists $c \in N(a)$ s.t. $c \not\sim (M{-}d_2)$. Thus, we can set $Q = M{-}d_2$ and we are done. □

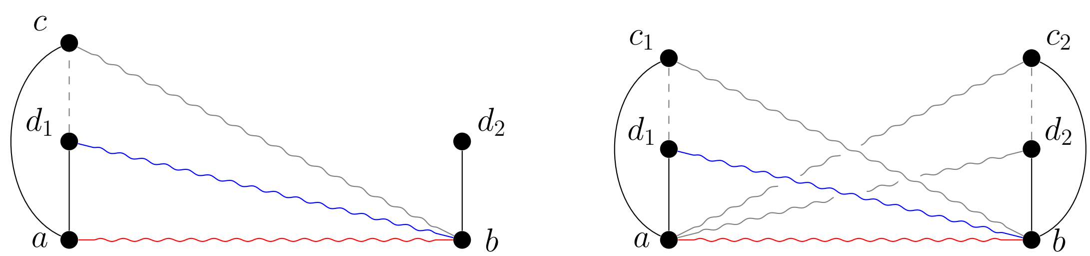

Figure 2.2: On the left we depict a single corner vertex c. On the right, Lemma 12 is applied to both dominating pair vertices a and b. The red path is a shortest a, b-path of length k. The blue path is Q within the proof. Dotted edges may not exist.

The remaining proofs will utilize the existence of corner vertices to resolve various properties of strict dominating pair graphs.

Lemma 13. *Under the hypotheses of Lemma 12, let P_1, P_2 be diametral d_1, d_2-paths containing b and a, respectively. Let c_1 and c_2 be corner vertices w.r.t. (P_1, a) and (P_2, b), respectively. If $c_1 \sim c_2$ then* $\text{diam}(G) \in \{3, 4\}$.

Proof. It is easy to check that $k \geq 2$. Notice that there exists a path $R = \langle a, c_1, c_2, b \rangle$ of length 3. If R is induced, then $k = 3$ and thus $\text{diam}(G) = k + 1 = 4$. If R is not induced, then $k < 3$. □

Next, we consider the effect that corner vertices have on the asteroidal number of the graph. To prepare for the theorem, we resolve general consequences of having a corner vertex that does *not* belong to a diametral pair.

Lemma 14. *Under the hypotheses of Lemma 12, let c be a corner vertex w.r.t. (Q, a) where Q is a diametral d_1, d_2-path that contains b. The following hold:*

1. *If R is a shortest c, b-path and $d_G(c, b) < k$, then $N[d_1] \cap R = \emptyset$.*

2. *If $G \setminus N[d_1]$ disconnects c from b, then (c, d_2) is a diametral pair. Moreover, if R' is any shortest c, d_2-path in G, then d_1 is adjacent to the first vertex on R' following c.*

Proof. Assume for the sake of contradiction that $v \in N[d_1] \cap R$. Since $v \neq c$ the walk $d_1 {-} R[v, b] {-} d_2$ has length strictly less than $k + 1$, a contradiction.

For the second half, let R' be a shortest c, d_2-path and suppose that $d_G(c, d_2) \leq k$. Since $R' {-} b$ is a c, b-walk, there exists w on $R' {-} b$ such that $w \in N[d_1]$. Note that $w \neq c$ by definition and $w \neq b$ since $k \geq 2$. The path $R'' = d_1 {-} R'[w, d_2]$ has length strictly less than $k + 1$, a contradiction. Thus (c, d_2) is diametral and R' has length $k + 1$. Note that the same argument shows that w cannot occur later than the first vertex on R' following c. □

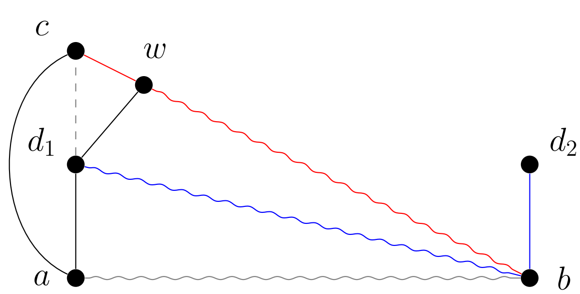

Figure 2.3: Depicting Lemma 14 where (c, d_2) is diametral. The dashed edge cannot exist. The blue path is Q and the red path is $R' \setminus \{d_2\}$.

We can see that Lemma 14 is symmetric with respect to a and b for a given dominating pair (a, b).

To simplify the following proof, we define new relationship notation. Let (a, b) be a dominating pair and let (d_1, d_2) be a diametral pair such that $d_1 \in N(a)$ and

$d_2 \in N(b)$. Let c be a corner vertex with respect to (P, a) where P is a diametral d_1, d_2-path that contains b. In that case, we say that $d_1 \prec^P c$.

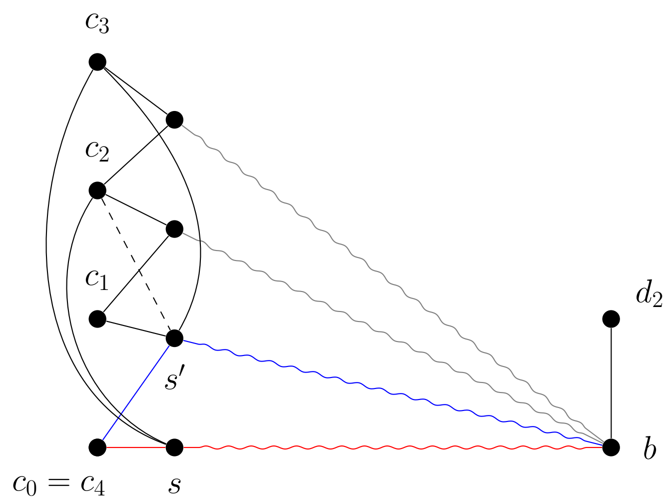

Figure 2.4: Depicting Lemma 15 where $c_3 \prec^{P_3} c_0$, so $h = 0$. The red path is $P_h \setminus \{d_2\}$ and the blue path is $P_{h'} \setminus \{d_2\}$.

Lemma 15. *Let G satisfy the hypotheses of Lemma 12 and let* $\text{diam}(G) > 4$. *There exist $c, c' \in N(a)$ with a shortest c, b-path P and a shortest c', b-path P' s.t.* $N[c] \cap P' = N[c'] \cap P = \emptyset$.

Proof. By Corollary 10, we let (d_1, d_2) be a diametral pair s.t. $d_1 \in N(a)$ and $d_2 \in N(b)$. We let $c_0 = d_1$ and P_0 be a diametral d_1, d_2-path that contains b. We will construct a sequence of distinct corner vertices $c_1, c_2, \ldots$ in $N(a)$, and diametral paths $P_1, P_2, \ldots$ such that P_i is a c_i, d_2-path and c_{i+1} is a corner vertex w.r.t. (P_i, a). Moreover, our sequence will satisfy the condition that for each c_i, set $N[c_i]$ meets every shortest c_{i+1}, b-path.

To be precise, given $c_0 \prec^{P_0} c_1 \prec^{P_1} \cdots \prec^{P_{i-1}} c_i \prec^{P_i}$ we let c_{i+1} be a corner vertex w.r.t. (P_i, a), as promised by Lemma 12, distinct from $c_0, \ldots, c_i$. If there is no corner vertex distinct from the earlier ones, the process terminates at the path P_i.

The step above alternates with the one we describe now, that of finding a path to add to $c_0 \prec^{P_0} c_1 \prec^{P_1} \cdots \prec^{P_{i-1}} c_i \prec^{P_i} c_{i+1}$. At this point there are two ways at which we might be done. If $d_G(c_{i+1}, b) < k$ then by the definition of a corner vertex and Lemma 14 we have $N[c_{i+1}] \cap P_i = \emptyset$ and $N[c_i] \cap P_{i+1} = \emptyset$, so we are done, letting $c = c_{i+1}$, $c' = c_i$, $P' = P_i \setminus \{d_2\}$, and $P = P_{i+1} \setminus \{d_2\}$. Also, if $N[c_i]$ does not meet every shortest c_{i+1}, b-path, then there exists a c_{i+1}, b-path R of length k in $G \setminus N[c_i]$. Now $c = c_{i+1}$, $c' = c_i$, $P' = P_i \setminus \{d_2\}$, and $P = R$ satisfy the conclusion of the lemma.

On the other hand if $d_G(c_{i+1}, b) = k$ and $N[c_i]$ does meet every shortest c_i, b-path, then by Lemma 12 there exists a diametral c_{i+1}, d_2-path P_{i+1} that contains b which we add to the end of our sequence.

If in our construction we never succeeded in producing the required c, c', P, P' then the construction must have terminated because we could not find a corner vertex c_{f+1} distinct from all the earlier c_i. Thus we have constructed

$$c_0 \prec^{P_0} c_1 \prec^{P_1} c_2 \prec^{P_2} \cdots \prec^{P_{f-1}} c_f \prec^{P_f} .$$

By Lemma 12 there *is* a corner vertex c_{f+1} w.r.t. (P_f, a) and so, by assumption, there exists h in the set $\{0, 1, \ldots, f-1\}$ such that $c_{f+1} = c_h$. Suppose that $h = f - 1$. Then we have $(c_{f-1} = c_h) \sim P_f$, a contradiction. Otherwise, we have $h < f - 1$. We will prove that this leads to a contradiction.

Let $P'_h = c_h {-} (P_{h+1} \setminus \{c_{h+1}\})$ and let s, s' be the first vertices in P_h, P'_h following c_h. We will prove by reverse-induction that for all $h + 1 < j \leq f$ that $c_j \sim s$ and $c_j \sim s'$. This holds for $j = f$ since $c_{f+1} = c_h$ and by construction $N[c_f]$ meets every shortest c_{f+1}, b-path.

Now suppose that $h + 1 < j < f$. By the inductive hypothesis $c_{j+1} {-} (P_h \setminus \{c_h\})$ is a diametral c_{j+1}, d_2-path, and thus, since $N[c_j]$ meets every diametral c_{j+1}, d_2-path at its second vertex, $c_j \sim s$. Similarly, since $c_{j+1} - (P'_h \setminus \{c_h\})$ is diametral, $c_j \sim s'$ (Fig. 2.4).

Finally, when $j = h+2$ we reach a contradiction. We must have $c_{h+2} \not\sim s'$ since c_{h+1} is a corner w.r.t. (P_{h+1}, a) and $s' \in P_{h+1}$. This contradiction establishes that the construction must have terminated through one of the conditions that give us appropriate c, c', P, P'. □

Lemma 15 is also symmetric with respect to either vertex in a given dominating pair. Applying the lemma twice, we find there are four vertices such that the removal of the closed neighborhood of any one of them does not disconnect the remaining three. These four vertices form an asteroidal set of size 4. Thus, we complete our main result:

Theorem 16. *Let G be a strict dominating pair graph. Either* $\mathrm{diam}(G) \in \{3, 4\}$ *or* $\mathrm{an}(G) \geq 4$.

We remark that the converse of the above theorem is not true. For example, consider the graph shown on the left in Fig. 2.1. If a pendant is added to the left-most black vertex, we obtain a graph that is HDP, has a dominating diametral path, has asteroidal number 4, and has diameter 4. Interestingly, a similar result was proven for directed path graphs that are non rooted path graphs by Gutierrez et al. [22], who showed that these graphs necessarily contain an asteroidal quadruple.

The class of strict dominating pair graphs is infinite. Fig. 2.5 gives more examples of graphs in this class. It is easy to construct infinitely many similar examples.

As stated by Corollary 10, strict dominating pair graphs have a diameter more tightly constrained to the distance between any two dominating pair vertices than HDP graphs in general. It is also easy to check that these graphs may not have diameter less than or equal to 2. No linear-time method to compute the diameter is

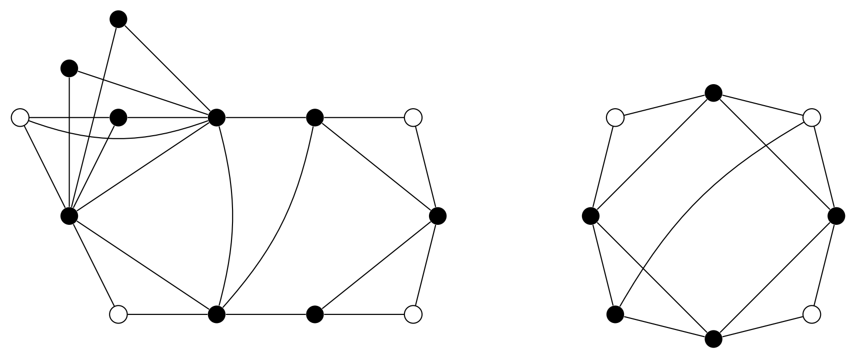

Figure 2.5: Strict dominating pair graphs. The graph on the left has $\mathrm{diam}(G) = 4$ and an asteroidal quadruple, whereas the right-hand graph has $\mathrm{diam}(G) = 3$ and $\mathrm{an}(G) = 3$.

known to exist for AT-free, HDP, or diametral path graphs even when a dominating pair is given. Hence, we have proven the following corollary concerning the complexity of diameter on this class of graphs.

Corollary 17. *Let G be a strict dominating pair graph with a given dominating pair. The diameter can be computed in linear-time.*

Proof. Let (a, b) be the given dominating pair. We perform a breadth-first-search to calculate $d_G(a, b)$. By Corollary 10, we have that $\mathrm{diam}(G) = d_G(a, b) + 1$. □

2.4 Cut Sets in HDP Graphs

In this section, we explore the structure of graphs that have dominating pairs and asteroidal triples, and show that dominating pair paths are necessarily "funnelled" through the common neighborhood of some pair of each asteroidal triple. Our intuition is that as asteroidal number increases, the placement of a dominating pair path is restricted such that there are never more than two distinct points of contact required to dominate an asteroidal set. We complete this section with an improved algorithm for the recognition of chordal HDP graphs.

2.4.1 Asteroidal Paths and Dominating Pairs

If a graph contains a dominating pair (a, b) and an asteroidal set S, then any a, b-path P dominates every vertex in S. We define notation in order to more easily refer to the outermost vertices in P that dominate vertices in S.

Definition 18. *Let (a, b) be a dominating pair, let S be an asteroidal set, and let P be an a, b-path. We denote by f^P (*resp. *ℓ^P) the first (*resp. *last) vertex of P that is adjacent to any vertex in S. These exist since P dominates S. We let $F^P = N(f^P) \cap S$ and $L^P = N(\ell^P) \cap S$. When necessary to distinguish, we will write F^P_S and L^P_S.*

Certainly $f^P \in C_N(F^P)$ and $\ell^P \in C_N(L^P)$. It is possible that $f^P = \ell^P$. With the following, we show that no such vertex is on any asteroidal path.

Proposition 19. *Let G contain an dominating pair (a, b) and an asteroidal set S, then if P is an a, b-path with $f^P = \ell^P$ then S is an asteroidal set in $G \setminus \{f^P\}$.*

Proof. Since there is only one vertex in P, *viz.* $f^P = \ell^P$, that is adjacent to any vertex in S, we have that f^P dominates S, and in particular is not on any asteroidal path for S. □

Next, we discuss the cardinality of F^P and L^P, for a given dominating pair path P.

Lemma 20. *Let G be HDP. Given a dominating pair (a, b), an a, b-path P and an asteroidal set S, then $|F^P| \geq 2$ or $|L^P| \geq 2$.*

Proof. If F^P or L^P have size at least 2, we are done. Otherwise, suppose $F^P = \{x\}$. If $L^P = \{x\}$ then $P[a, f^P]–x–P[\ell^P, b]$ is an a, b-path not dominating any vertex in $S \setminus \{x\}$. Suppose then w.l.o.g. that $L^P = \{y\}$, with $y \neq x$. Pick $z \in S \setminus \{x, y\}$. Since S is an asteroidal set, there exists an x, y-path P' that does not dominate z. Now

$P[a, f^P]{-}P'{-}P[\ell^P, b]$ is an a, b-walk from which we can extract an a, b-path that does not dominate z, a contradiction. □

Pržulj, Corneil, and Köhler [24] investigated a subclass of HDP graphs called *frame HDP*. Below, we give a definition of a frame HDP graph.

Definition 21. *A frame hereditary dominating pair (frame HDP) graph G is a hereditary dominating pair graph with an asteroidal triple T such that all vertices of G are on some asteroidal path with endpoints in T.*

In particular, Pržulj, Corneil, and Köhler explored the location of dominating pairs in frame HDP graphs and proved that such graphs have $\text{diam}(G) \leq 5$. They showed that every dominating pair satisfies strong constraints on the location of the endpoints relative to the fixed asteroidal triple. Lemma 20 generalizes this by putting constraints on all paths between any dominating pair of vertices.

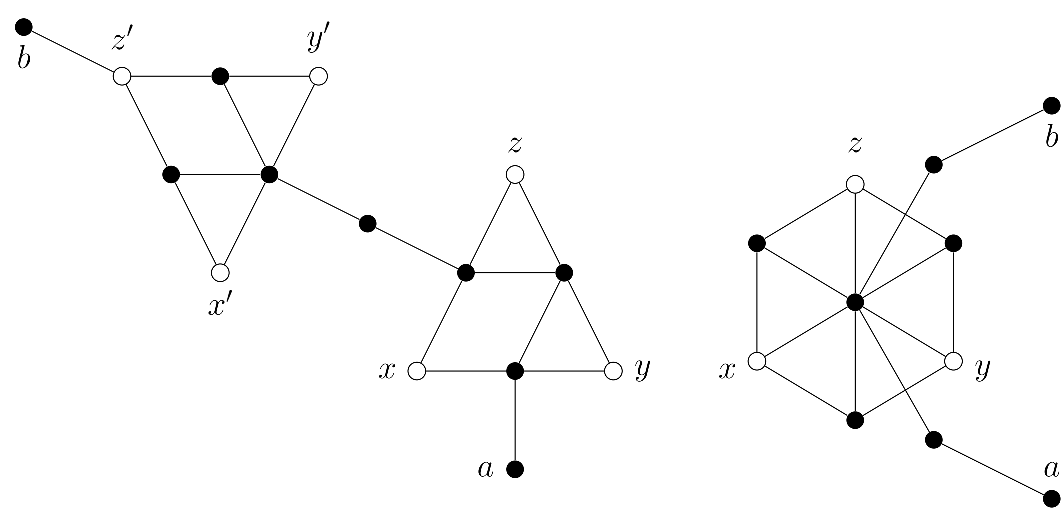

Figure 2.6: Some HDP graphs that are not frame HDP. In each case vertices a, b are not on any asteroidal path for $T = \{x, y, z\}$ or $T = \{x', y', z'\}$.

An important result about the structure of HDP graphs is directly implied by Lemma 20:

Corollary 22. *HDP graphs do not contain an asteroidal triple with* 3*-spread.*

Proof. Let G be HDP. Given an asteroidal triple T with 3-spread, let $H = G \setminus (C_N(x,y) \cup C_N(y,z) \cup C_N(x,z))$. H has a dominating pair (a,b) and contains T. Let P be an arbitrary a,b-path. Then, by Lemma 20 one of f^P, ℓ^P is in the common neighborhood of some pair from T, a contradiction. □

Diametral path graphs may contain asteroidal triples with arbitrarily large spread. See, for instance, the right-hand graph in Fig. 2.1.

In Lemma 20, we proved that given a dominating pair (a,b) and an asteroidal triple T, every a,b-path passes through the common neighborhood of some pair from T. In fact, we will show in Theorem 26 that such a pair can be chosen uniformly for all a,b-paths. Before we can prove such a theorem, we prove two useful lemmas. For now, we apply Definition 18 with respect to asteroidal sets of cardinality 3 (i.e., asteroidal triples). Later, we will generalize these lemmas to graphs with greater asteroidal number.

Lemma 23. *Let G be HDP. Given a dominating pair (a,b), an AT $\{x,y,z\} = T$, and a,b-paths P and P', we have $F^P \cup L^{P'} = F^{P'} \cup L^P = T$.*

Proof. By Lemma 20, either $|F^P| \geq 2$ or $|L^P| \geq 2$. W.l.o.g., suppose that $F^P = \{x,y\}$. If $L^{P'}$ contains z, then we are done. First we will show that $F^P \cup L^{P'} = T$. Otherwise, since $L^{P'}$ is not empty, we may suppose that $L^{P'}$ contains x. The walk $P[a, f^P]{-}x{-}P'[\ell^{P'}, b]$ contains an a,b-path that does not dominate z, a contradiction.

Now we will show that $F^{P'} \cup L^P = T$. By the first paragraph, applied with $P = P'$, we know that $z \in L^P$. If $\{x,y\} \subseteq F^{P'} \cup L^P$ we are done. Otherwise, w.l.o.g. $x \notin F^{P'} \cup L^P$. If $y \in F^{P'}$ then let R be an asteroidal y,z-path. The walk $P'[a, f^{P'}]{-}R{-}P[\ell^P, b]$ contains an a,b-path that does not dominate x, a contradiction. Finally, the only remaining possibility is that $F^{P'} = \{z\}$, in which case $P'[a, f^{P'}]-z-P[\ell^P, b]$ contains an a, b-path that does not dominate x, a contradiction. □

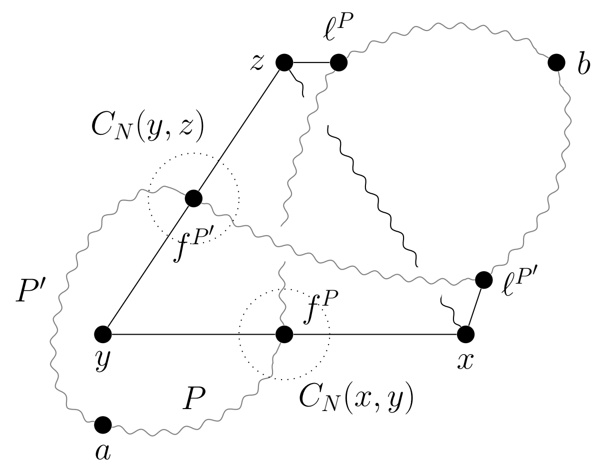

Figure 2.7: Setup of the first part of Lemma 23. The a, b-walk $P[a, f^P]-x-P[\ell^{P'}, b]$ does not dominate z.

At this stage, Lemmas 20 and 23 give us a strong understanding of how a dominating pair path dominates asteroidal triples. Next, we give a lemma that describes the orientation of two or more a, b-paths that meet the same common neighborhood of some pair in an asteroidal triple.

Lemma 24. *Let G be HDP. Given a dominating pair (a, b), an AT $\{x, y, z\} = T$, and two a, b-paths P and P', it cannot be that $F^P = L^{P'}$ unless $F^P = L^{P'} = T$.*

Proof. Suppose that $F^P = L^{P'} \neq T$. W.l.o.g. F^P contains x but not z. Then let $Q = P[a, f^P]-x-P'[\ell^{P'}, b]$. From the walk Q, we can extract a path that does not dominate z, a contradiction. □

We prove an important lemma that will be useful for proving the proceeding theorem.

Lemma 25. *Let G be HDP. Given dominating pair (a, b), an asteroidal triple $T = \{x, y, z\}$, and two a, b-paths P, P', there exists a pair in T s.t. P and P' meet its common neighborhood.*

Proof. First note that if either path meets $C_N(T)$, then by Lemma 20 the result holds.

Case 1. Suppose that one of the paths, w.l.o.g. P, has F^P and L^P disjoint, so in particular one of them is a singleton. W.l.o.g. $F^P = \{x, y\}$ and $L^P = \{z\}$. By Lemma 23, $F^{P'}$ contains $\{x, y\}$ and we are done.

Case 2. Otherwise, each of $F^P, L^P, F^{P'}$ and $L^{P'}$ have size 2. Then two of them and equal and by Lemma 20 they cannot be contained on the same path; therefore, we are done. □

We are prepared to prove a major theorem that describes the structure of dominating pair paths in asteroidal triples in HDP graphs.

Theorem 26. *Let G be HDP. Given a dominating pair (a, b) and an asteroidal triple $T = \{x, y, z\}$, there exists a pair $D_{a,b} \subseteq T$ s.t. all a, b-paths meet $C_N(D_{a,b})$.*

Proof. Suppose for the sake of contradiction that P, P', and P'' are a, b-paths avoiding $C_N(x, y), C_N(x, z)$, and $C_N(y, z)$, respectively. Consider $I = F^P \cap F^{P'} \cap F^{P''}$. Everything in $J = T \setminus I$ is contained, by Lemma 23, in $J = L^P \cap L^{P'} \cap L^{P''}$. Thus one of I, J has size at least 2, a contradiction. □

Theorem 26 is important because it shows that an example like the one shown on the left of Fig. 2.8 may not occur. The unique structure of asteroidal triples that are allowed in HDP graphs shows that if (a, b) is a dominating pair in such a graph and $D_{a,b}$ corresponds to some asteroidal triple T then either $C_N(D_{a,b}) \cap (a, b)$ is non-empty or $C_N(D_{a,b})$ is a cut set in G.

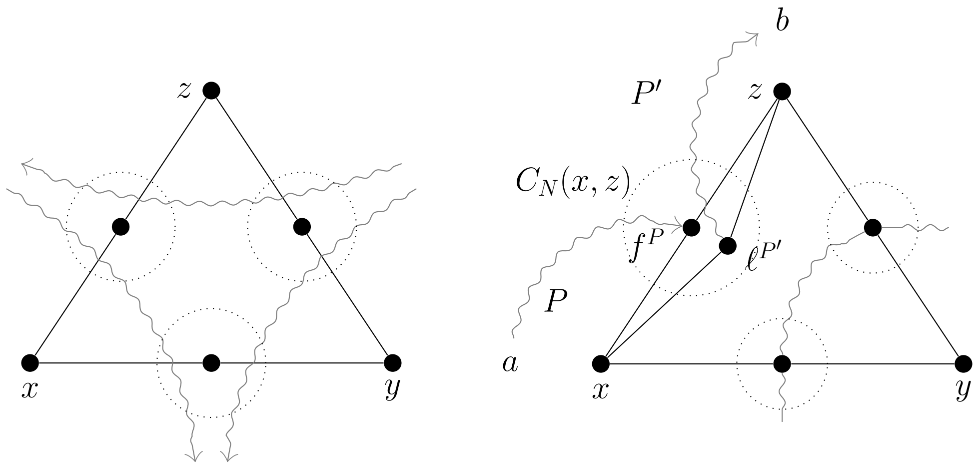

Figure 2.8: On the left, any pair of a, b-paths shown satisfy Lemma 25. But, Theorem 26 is contradicted. Dotted circles represent the the common neighborhoods of pairs in the AT $\{x, y, z\}$. On the right, we detail that the left graph contains an a, b-walk $P[a, f^P]{-}z{-}P'[\ell^{P'}, b]$ that does not dominate y.

We have strengthened Lemma 20 with Theorem 26. Recall that they apply to every asteroidal triple in an asteroidal set S. The next lemma generalizes Lemma 23 to asteroidal sets of arbitrary size.

Lemma 27. *Let G be HDP. Given an asteroidal set S, a dominating pair (a, b), and a, b-paths P and P', we have $F^P \cup L^{P'} = F^{P'} \cup L^P = S$.*

Proof. Suppose that $F_S^P \cup L_S^{P'} \neq S$. By Lemma 20, one of F_S^P and $L_S^{P'}$ has size at least 2. W.l.o.g. we assume $|F_S^P|$ is at least 2. Therefore, we can pick $x \in F_S^P$ and $y \in L_S^{P'}$ with $x \neq y$. By assumption, there exists $z \notin F_S^P \cup L_S^{P'}$. Now we will consider $F^P_{\{x,y,z\}}$ and $L^{P'}_{\{x,y,z\}}$. Since $x \in F_S^P$ and $y \in L_S^{P'}$, we have $f_S^P = f^P_{\{x,y,z\}}$ and $\ell_S^{P'} = \ell^{P'}_{\{x,y,z\}}$. By assumption, z belongs to neither of $F^P_{\{x,y,z\}}$ nor $L^{P'}_{\{x,y,z\}}$. This contradicts Lemma 23. □

Next, we make a more general statement regarding all dominating pair paths between a given dominating pair and asteroidal sets of any size in HDP graphs.

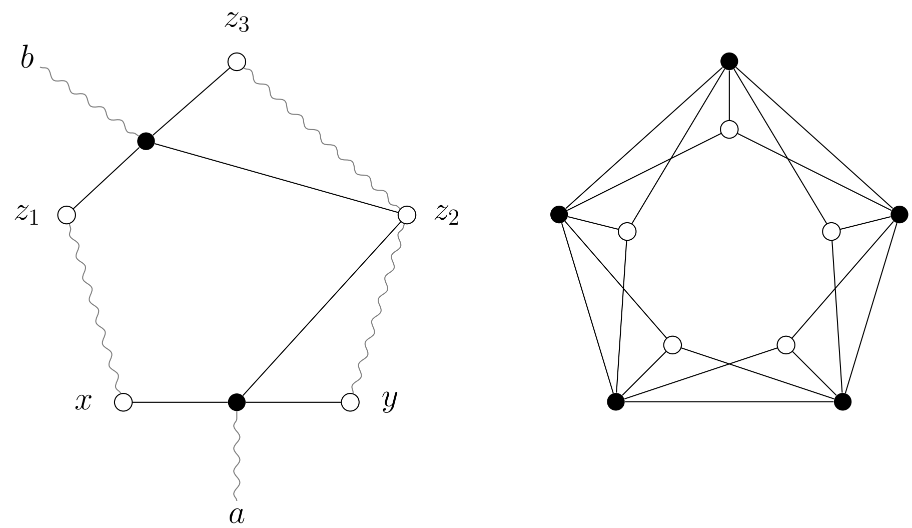

Figure 2.9: Given a dominating pair (a, b) and letting P be an a, b-path on the left, we have an HDP graph with $F^P \cup L^P = S$. On the right, we show an example with asteroidal number 5. White vertices denote the asteroidal set.

Theorem 28. *Let G be HDP. Given an asteroidal set S and a dominating pair (a, b), let $F = \bigcap_P F_S^P$ and $L = \bigcap_P L_S^P$ for all a, b-paths P. Then $F \cup L = S$.*

Proof. Suppose otherwise. There exists $x \in S$ where $x \notin F$ and $x \notin L$. Then, there exist a, b-paths P', P'' such that $x \notin F_S^{P'}$ and $x \notin L_S^{P''}$. This contradicts Lemma 27. □

Consider Fig. 2.9. In both graphs, the white vertices denote the asteroidal set S. On the left graph we depict Theorem 28 where $F = \{x, y, z_2\}$ and $L = \{z_1, z_2, z_3\}$. On the right graph, we show a specific example of the more symbolic depiction shown on the left.

In general, HDP graphs become inherently more dense as asteroidal number increases. An interesting and extreme case of Theorem 28 is where $F = S$ (or, $L = S$). Then, it trivially holds that $F \cup L = S$. Consequently, $C_N(F)$ dominates S despite $C_N(F)$ being a cut set in G. Observe that $C_N(F)$ does not establish S in this case, so the removal of $C_N(F)$ leaves a connected, induced subgraph that contains the asteroidal set S, and this follows by Proposition 19. We demonstrate this case in the right graph of Fig. 2.6.

2.4.2 On Networks and Faster Recognition of Chordal HDP Graphs

With respect to application of HDP graphs in critical systems or ad hoc networks, Theorem 28 poses a problem. A cut vertex or cut set is naturally a point of weakness in a network. Therefore, if one is interested in utilizing dominating pairs, the inclusion of an asteroidal triple may necessitate the reinforcement of articulation vertices in some manner. Certainly, such a restriction gives us greater control in regards to the algorithmic complexity of certain problems in HDP graphs. The notion of spread, in particular, is a useful algorithmic tool.

We present a method for faster recognition of chordal HDP graphs from a previous best run-time of $O(n^7)$ in [24]. A complete set of forbidden subgraphs for the class of chordal HDP graphs is shown in their paper and reproduced in Fig. 2.10. The forbidden subgraphs have an asteroidal triple with 3-spread. Therefore, a faster algorithm for recognition is apparent.

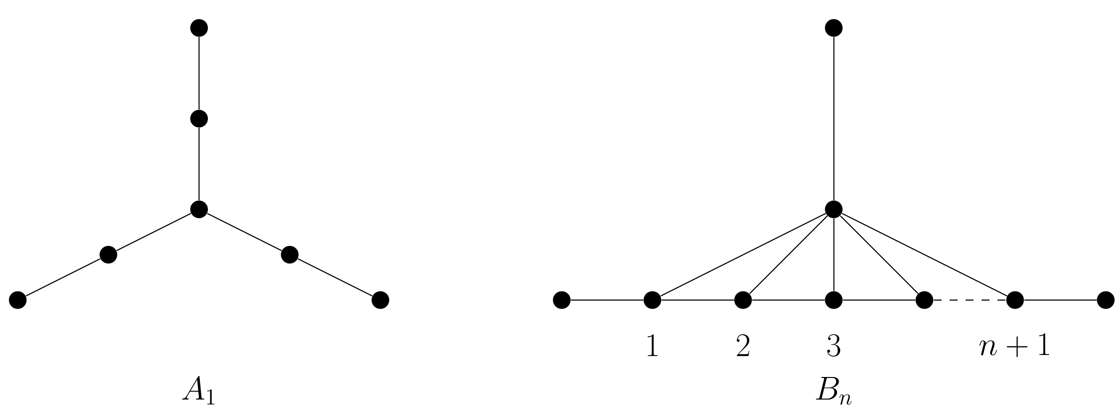

Figure 2.10: Forbidden induced subgraphs for Chordal HDP graphs.

Theorem 29. *Chordal HDP graphs are exactly the graphs that are chordal and have no asteroidal triples with* 3*-spread. In particular, chordal HDP graphs can be recognized in* $O(n^{3.82})$.

Proof. If G contains an asteroidal triple with 3-spread, then by Corollary 22, G is not HDP. On the other hand, if G contains no asteroidal triple with 3-spread, then in particular it does not contain any of the forbidden subgraphs that characterize chordal HDP graphs (see Fig. 2.10), and hence is chordal HDP.

Iterating through all asteroidal triples in a graph G requires time $O(n^{2.82})$ [20]. Additionally, checking whether an asteroidal triple has 3-spread requires time $O(n)$ and can be accomplished as follows. Let $T = \{x, y, z\}$ be any asteroidal triple in G. We remove the sets $C_N(x, y)$, $C_N(y, z)$ and $C_N(x, z)$ from G to form subgraph H. We check that T is an asteroidal triple in H, which requires linear time. If so, then T has 3-spread. Lastly, it is well-known that checking that a graph is chordal is linear [14]. Thus, the total time-complexity is $O(n^{3.82})$. □

2.5 Properties of HDP Graphs

In this section, we closely study the internal structure of graphs that do not have an asteroidal triple with 3-spread. Then, we show how the structure is further restricted if the graph is HDP. For such graphs, we also show that an asteroidal quadruple may apply a constraint on diameter. Then, we investigate findings relating the diameter of HDP graphs to vertices in *polar pairs*.

2.5.1 The Internal Structure of HDP Graphs

Recall that any three vertices in an asteroidal set form an asteroidal triple. We utilize the hereditary nature of asteroidal sets to prove the following:

Proposition 30. *Let G contain no AT with 3-spread and let S be an asteroidal set. If* $\mathrm{an}(G) = t = |S|$*, then there are at least $t-2$ pairs $A \subseteq S$ s.t. $C_N(A) \neq \emptyset$.*

Proof. Choose three vertices $\{x, y, z\} \subseteq S$. They form an asteroidal triple T for which at least one pair has a non-empty common neighborhood (or we contradict the definition of G).

Case 1: Exactly one pair $A \subseteq T$ has a non-empty common neighborhood. W.l.o.g. let $A = \{x, y\}$. We remove either x or y from S.

Case 2: Exactly two pairs in T have a non-empty common neighborhood. W.l.o.g. let $A_1 = \{x, y\}$ and $A_2 = \{x, z\}$ s.t. $C_N(A_1) \neq \emptyset$ and $C_N(A_2) \neq \emptyset$. We remove y and z from S.

Case 3: All three pairs in T have a non-empty common neighborhood. We remove x, y, z from S.

We repeat the above cases until no asteroidal triples remain in S. After k vertices have been removed, we have $t-k$ vertices in S remaining, no asteroidal triple of which has three pairs with non-empty neighborhoods. Thus, if $t-k \geq 3$, we choose any remaining $T \subseteq S$ and repeat one of the above cases. With each step, we remove either 1, 2, or 3 vertices from G and find just as many pairs in S with non-empty neighborhoods in G. After $t-2$ vertices have been removed, we have found $t-2$ pairs in S with non-empty neighborhoods in G. □

Next, we are interested in analyzing structure that is internal to any given asteroidal triple.

Lemma 31. *Given an asteroidal triple $T = \{x, y, z\}$ that does not have 3-spread, suppose that $C_N(x, y) \neq \emptyset$. Let $P_{x,z}$ and $P_{y,z}$ be an asteroidal x, z-path and an asteroidal y, z-path, respectively. Let $p \in P_{x,z} \setminus \{x\}$ and $q \in P_{y,z} \setminus \{y\}$. Then all p, q-paths meet $C_N(x, y)$ or $N[z]$.*

Proof. Suppose otherwise. Let P be a p,q-path that avoids $C_N(x,y)$ and $N[z]$. In particular, $p,q \notin N[z]$, i.e., $d_G(z,p) \geq 2$ and $d_G(z,q) \geq 2$. Consequently, $P_{x,z}$ has length 3 or more and avoids $N[y]$ while $P_{y,z}$ has length 3 or more and avoids $N[x]$. Thus, $W = P_{x,z}[x,p]{-}P{-}P_{y,z}[q,y]$ is an x,y-walk that avoids $N[z]$ and $C_N(x,y)$. From W we can extract an asteroidal x,y-path $P_{x,y}$ that in particular has length at least 3. The paths $P_{x,z}, P_{y,z}, P_{x,y}$ certify that T has 3-spread, a contradiction. □

Another way of phrasing the conclusion of Lemma 31 is that every induced x,y-path of length 3 or more is non-asteroidal.

Notice that Lemma 31 is at its "weakest" when asteroidal number is 3 because there are fewer possible candidates for z for a given pair of asteroidal vertices x,y. Then, there is less restriction on the placement of paths between the asteroidal paths that establish an asteroidal triple. We will show that Lemma 31 can be strengthened on HDP graphs.

Lemma 32. *Let G be HDP. Given an AT $\{x,y,z\}$, a dominating pair (a,b) and an a,b-path P, let $P_{x,z}, P_{y,z}$ be an asteroidal x,z-path and an asteroidal y,z-path, respectively. Suppose that $F^P \supseteq \{x,y\}$ and $L^P \supseteq \{z\}$. Then, every vertex in $P_{x,z} \cup P_{y,z}$ is adjacent to f^P or $N[\ell^P]$.*

Proof. W.l.o.g. suppose for the sake of contradiction that there exists $p \in P_{x,z}$ where $p \not\sim f^P$ and $p \not\sim N[\ell^P]$. Notice that $p \not\sim y$ by the definition of the asteroidal path $P_{x,z}$. Let $R = N[\ell^P] \cup \{y\}$. There exists an a,b-walk $P[a,f^P]{-}P_{y,z}{-}P[\ell^P,b]$ that must dominate p.

We claim that neither $P[a,f^P]$ nor $P[\ell^P,b]$ are adjacent to p. Suppose $v \in P[a,f^P]$ is adjacent to p. Obviously $p \neq f^P$. We have that $P[a,v]{-}P_{x,z}[p,z]{-}P[\ell^P,b]$ does not dominate y, a contradiction. Suppose instead that $v' \in P[\ell^P,b]$ is adjacent to p.

Obviously $p \neq \ell^P$. We have that $P[a, f^P]-P_{x,z}[x,p]-P[v', b]$ does not dominate y, a contradiction. We have proven the claim.

So, it is necessary that $p \sim P_{y,z}$. In particular, $p \sim P_{y,z} \setminus R$. Let p be adjacent to $q \in P_{y,z} \setminus R$. The path $\langle p, q \rangle$ (or in the case that $p = q$, the path $\langle p \rangle$) avoids $C_N(x, y)$ and $N[z]$, a contradiction to Lemma 31. $\square$

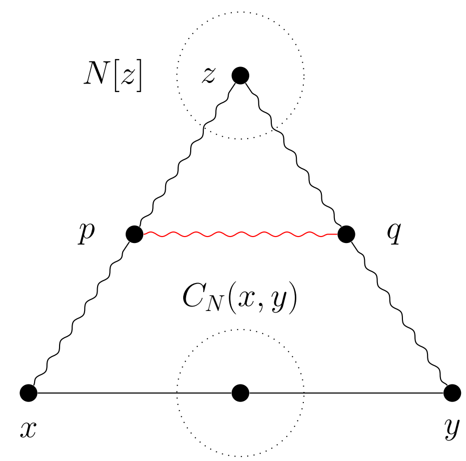

Figure 2.11: This graph follows the proof of Lemma 31. The red path may not exist since it does not meet $N[z]$ or $C_N(x, y)$.

2.5.2 Asteroidal Quadruples and Diameter Bounds

Previously in Sect. 2.3 we showed that asteroidal quadruples are a key structural attribute of strict dominating pair graphs with high diameter (see Theorem 16). Interestingly, the existence of an asteroidal quadruple provides the opportunity for diameter restriction in an HDP graph if a shortest dominating pair path meets it in a specific way. To assist in our proof, for an asteroidal triple $\{x, y, z\}$ let $T^z_{x,y}$ denote an asteroidal x, y-path that avoids $N[z]$.

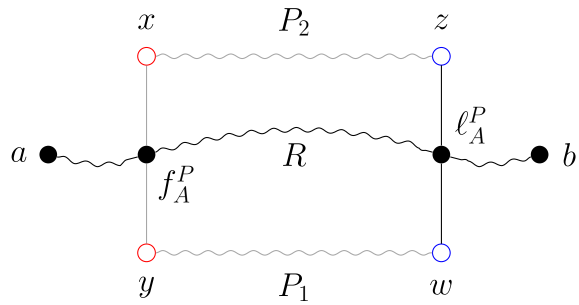

Figure 2.12: Portraying the proof of Theorem 33. The red vertices are in $F_A^P \setminus S$ and the blue vertices are in $L_A^P \setminus S$. The gray paths depict W.

Theorem 33. *Let G be HDP. For some dominating pair (a, b), an asteroidal set A, and a shortest a, b-path P, let $S = F_A^P \cap L_A^P$. If $F_A^P \setminus S \geq 2$ and $L_A^P \setminus S \geq 2$ then* $\operatorname{diam}(G) \leq 12$.

Proof. Suppose for the sake of contradiction that $\operatorname{diam}(G) > 12$. Let $\{w, x, y, z\} \subseteq A$ be an asteroidal set formed of two vertices $x, y \in F_A^P \setminus S$ and two vertices $w, z \in L_A^P \setminus S$. By Lemma 32, for some asteroidal paths $T_{x,z}^y$ and $T_{y,z}^x$ we have that all vertices in $T_{x,z}^y \cup T_{y,z}^x$ are adjacent to f_A^P or $N[\ell_A^P]$. Therefore, any shortest f_A^P, ℓ_A^P-path has length at most 4. In particular, $R = P[f_A^P, \ell_A^P]$ has length at most 4.

We will show that $d_G(a, f_A^P) \geq 4$ or $d_G(b, \ell_A^P) \geq 4$. Suppose otherwise. Let $R_a = P[a, f_A^P]$ and $R_b = P[\ell_A^P, b]$. The paths R_a, R_b are shortest paths with lengths at most 3. The walk $R_a{-}R{-}R_b$ contains an a, b-path with length at most 10. By Corollary 10 we have that $\operatorname{diam}(G) \leq 12$, a contradiction. W.l.o.g. suppose that $d_G(a, f_A^P) \geq 4$.

We will show that every induced $T_{w,y}^z$ and every induced $T_{z,x}^w$ avoid $N[a]$. Suppose not. W.l.o.g. let U be an induced $T_{w,y}^z$ that meets $N[a]$. Let u be the first vertex in U that is adjacent to a. The a, b-walk $a{-}U[u, w]{-}R_b$ must dominate y and this is only possible if $u \in N[y] \cap U$ (recall that U is an induced w, y-path and that R_b

cannot be adjacent to y because $y \notin L^P_{\{w,x,y,z\}}$). There exists a walk $\langle a, u, y, f^P_A \rangle$ of length at most 3, a contradiction to $d_G(a, f^P_A) \geq 4$. We have proven the claim. Let P_1 and P_2 be an induced $T^z_{w,y}$ and an induced $T^w_{z,x}$, respectively, respectively, that each avoid $N[a]$. Let $M = \langle y, f^P_A, x \rangle$ and let $W = P_1 - M - P_2$. See Fig. 2.12. Note that M clearly avoids $N[a]$ since $d_G(a, f^P_A) \geq 4$.

In order to complete the prove of the theorem, it suffices to show that $T' = \{a, w, z\}$ is an asteroidal triple that has 3-spread, contradicting our assumption that G is HDP. W is a concatenation of asteroidal paths P_1 and P_2 that avoid $C_N(w, z)$ together with a path M that avoids $C_N(w, z)$ (i.e. M avoids $C_N(w, z)$ because $f^P_{\{w,x,y,z\}} \notin C_N(w, z)$ and the vertices of A are non-adjacent by the definition of an asteroidal set). This fact together with $W \cap N[a] = \emptyset$ implies that a w, z-path J_1 of length 3 or more that avoids $N[a]$ can be extracted from W. An a, z-path J_2 of length 3 or more that avoids $N[w]$ can be extracted from the walk $P[a, f^P_A] - P_2$. A similar path J_3 that avoids $N[z]$ can be extracted from the walk $P[a, f^P_A] - P_1$. The paths J_1, J_2, and J_3 establish that T' has 3-spread, a contradiction. □

Theorem 33 can also be stated as follows: The diameter of an HDP graph is less than or equal to 12 if each half of an asteroidal quadruple is uniquely contained in F^P and L^P for any shortest a, b-path P. Moreover, we can easily see that the hypotheses of Theorem 33 hold for any HDP graph containing an asteroidal quadruple A where any three vertices in A have an empty common neighborhood.

Among the literature, Theorem 33 provides yet another sufficient condition relating asteroidal sets to diameter restriction in an HDP graph. To wit, it has also been shown [24] that a frame HDP graph G has $\text{diam}(G) \leq 5$.

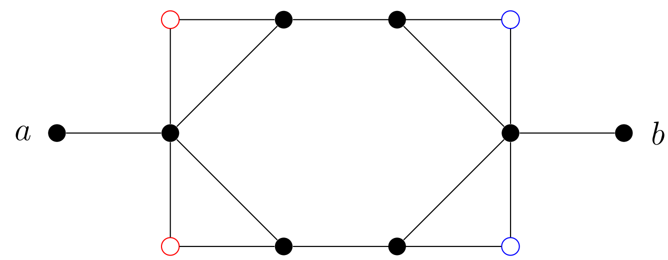

Figure 2.13: An HDP graph having diam$(G) = 5$. The unfilled vertices form an asteroidal quadruple A. For any a, b-path P, the red vertices are F_A^P and the blue vertices are L_A^P. If a pendant vertex p is added to b (thus increasing the diameter), then the red vertices along with p are an asteroidal triple with 3-spread. A similar technique is employed in the proof of Theorem 33.

2.5.3 On Maximum Eccentricity

In this subsection, we present additional distance-related findings beyond those given in Sect. 2.3. An especially strong property of a graph is to have a so-called *polar pair*. The following definition is essentially equivalent to what is originally given in [8].

Definition 34. *A pair (X, Y), where $X, Y \subseteq V$, is* polar *if $X \cap Y = \emptyset$ and a pair of vertices x, y is a dominating pair if and only if exactly one of them is in each of X and Y.*

We will present some diameter results for any graph G having a polar pair. The main result of this subsection (Theorem 37) shows that a dominating pair vertex satisfying a special property with respect to a polar pair has eccentricity equal to diam(G).

We remark that AT-free graphs having diam$(G) \geq 4$ and graphs with a dominating pair having diam$(G) \geq 5$ are certain to contain a polar pair [8, 24]. Moreover, such a pair can be computed in linear time if G is AT-free [7]. At low diameter, the vertices of dominating pair vertices cannot be partitioned in such a way as to satisfy Definition

34. For example, it is easily verified that the cycle C_5, which is AT-free, does not have a polar pair. The following preliminary result is obtained from [8] (see their Theorem 4.3).

Lemma 35. [8] *In every connected AT-free graph some dominating pair is a diametral pair.*

Our main results rely on certain nice properties provided by polar pairs.

Lemma 36. *Let G be a graph having a polar pair (X, Y). W.l.o.g. let A be a maximal clique in X. Then, A contains some vertex a where $\text{ecc}(a) = \text{diam}(G)$.*

Proof. Suppose otherwise. By Lemma 35 and Definition 34, some pair $(d_1, d_2) \in (X, Y)$ is diametral. W.l.o.g. let $D_1 \subseteq X$ be a maximal clique containing d_1. If $d_1 \in A \cap D_1$ then $a := d_1$ and we are done. Thus, there exists a vertex $x \in A$ such that $x \not\sim d_1$, otherwise we contradict the maximality of A. We assume that x cannot be set to a, otherwise we are done.

The pair (d_1, d_2) is diametral but (x, d_2) is not because $\text{ecc}(x) < \text{diam}(G)$. So, $\text{diam}(G) > d_G(x, d_2)$. Let $P = \langle x = v_0, v_1, \ldots, v_{k-1}, v_k = d_2 \rangle$ be a shortest x, d_2-path. Since (x, d_2) is a dominating pair, P must dominate d_1. Since $x \not\sim d_1$, we have that $d_1 \sim v_i$ for some $i > 0$. Let $Q = \langle v_i, \ldots, d_2 \rangle \subset P$. Any shortest path extracted from $d_1 - Q$ has length less than or equal to $d_G(x, d_2)$, a contradiction to the fact that $d_G(d_2, d_2) = \text{diam}(G)$. $\square$

It is known that linear diameter computation of an AT-free graph is unlikely due to an existing reduction to computing a simplicial vertex in a general graph [9]. Instead, we consider a special dominating pair vertex, that is not guaranteed to exist, but which can be used as a "shortcut" to fast diameter computation. We are prepared to prove the main result of this section.

Theorem 37. *Let G be a graph with* $\operatorname{diam}(G) > 4$ *and let* (X, Y) *be a polar pair in* G. *Every vertex* a *where* $N[a] \subseteq X$ *or* $N[a] \subseteq Y$ *has* $\operatorname{ecc}(a) = \operatorname{diam}(G)$.

Proof. W.l.o.g. let $x \in X$ have $N[x] \subseteq X$. For the sake of contradiction suppose that $\operatorname{ecc}(x) < \operatorname{diam}(G)$. Let $y \in Y$ and let $d_G(x, y) = k$. Note that $k < \operatorname{diam}(G)$. Any shortest x, y-path P must meet $N(x)$. Let $v = P \cap N(x)$. Notice that $v \in X$ and that $d_G(y, x) = d_G(y, v) + 1$.

The pair (v, y) is a dominating pair by definition of (X, Y). By Lemma 9, there exists pair (d_1, d_2) such that $d_1 \in N[v]$ and $d_2 \in N[y]$. The path $R = P \setminus \{x\}$ is a shortest v, y-path of length $k - 1$. If $d_G(d_1, d_2) = k$ then P (which also has length k) is diametral and so $\operatorname{ecc}(x) = \operatorname{diam}(G)$, a contradiction. Since $d_G(d_1, d_2)$ may not exceed the length of $d_1{-}R{-}d_2$, we have that $d_G(d_1, d_2) = k + 1 = \operatorname{diam}(G)$.

We will show that $\operatorname{ecc}(v) < \operatorname{diam}(G)$. Suppose not. For some vertex b we have that (v, b) is diametral and thus $d_G(v, b) = k + 1$. Recall that R is a dominating v, y-path of length $k - 1$. In particular, $b \sim R$. Let t be the first vertex in R that is adjacent to b. Consequently, $R[v, t]{-}b$ has length at most k, a contradiction to the value of $d_G(v, b)$. We have shown that $\operatorname{ecc}(v) < \operatorname{diam}(G)$.

The set $\{x, v\} \subseteq X$ is a clique on 2 vertices. By Lemma 36, since for each vertex $u \in \{x, v\}$ we have that $\operatorname{ecc}(u) < \operatorname{diam}(G)$, this clique is not maximal. There exists a clique $\{v, w, x\} \subseteq X$ such that $\operatorname{ecc}(w) = \operatorname{diam}(G)$ to satisfy Lemma 36. Let (w, d) be a diametral pair. Let Q be a shortest x, d-path. The length of Q is k, otherwise $w{-}Q$ has length less than $k+1$, a contradiction to the value of $d_G(w, d)$. Let $u = Q \cap N(x)$. Note that (u, y) is a dominating pair. See Fig. 2.14 for this stage of the proof.

Let $D_1 = Q \setminus \{x\}$ and let D_2 be a shortest d, y-path. Of course, D_1 has length $k - 1$. Since we have shown that $\operatorname{diam}(G) = k + 1$, path D_2 has length at most $k + 1$. Moreover, $D_1{-}D_2$ is a dominating u, y-walk. In particular, this walk dominates w.

Suppose that $w \sim D_1$. Let $w \sim u$. The path $w{-}D_1$ has length k, a contradiction to the fact that $d_G(w, d) = k + 1$. A similar contradiction occurs for any $p \in D_1 \setminus \{x\}$ where $w \sim p$. Therefore, $w \not\sim D_1$ and it is necessary that $w \sim D_2$.

Let f be the first vertex in D_2 that is adjacent to w. Recall that D_2 has length at most $k{+}1$. If $f \notin N[y]$ then the path $D_2[d, f]{-}w$ has length at most k, a contradiction to the value of $d_G(w, d)$. Thus, $f \in N[y]$. The walk $\langle y, f, w, x\rangle$ has length at most 3. It follows that $k \leq 3$ and $\mathrm{diam}(G) \leq 4$, a contradiction. □

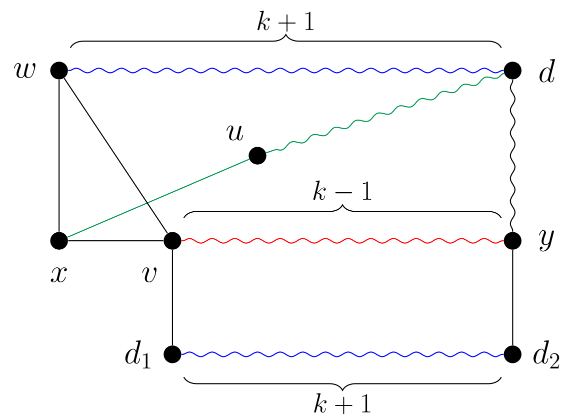

Figure 2.14: Depicting the proof of Theorem 37. The red path is R in the proof, the green path is Q in the proof, and the blue paths are diametral.

It is easy to see that a vertex a satisfying Theorem 37 can be computed via comparing edge incidence between X, Y, and G. If (X, Y) can be computed in linear-time, as it can be when G is AT-free and has $\mathrm{diam}(G) \geq 4$ [7], then a provides a special case for fast diameter computation.

Chapter 3

Polar Targets

A dominating pair in a graph G is a pair of vertices s, t such that every path connecting them dominates the graph. It is well-known, for instance, that any asteroidal triple-free graph contains a dominating pair. A much stronger property of a graph is that there are two subsets $S, T \subseteq V$ such that the dominating pairs in G are exactly all pairs taking one vertex from S and one vertex from T. Such a pair of subsets is called a *polar pair*. For instance, polar pairs are known to exist in asteroidal triple-free graphs with sufficiently high diameter.

In this chapter, we introduce the notion of a *polar target*, a collection of subsets $T_1, T_2, \ldots, T_k \subseteq V$ such that D is a dominating target of size k (i.e. any connected superset of D dominates G) if and only if $|D \cap T_i| = 1$ for all i. Our main contribution is to prove that if G has a dominating 4-distant asteroidal triple then it has a polar target of size 3. Along the way, we develop strong results about the possible interactions between dominating targets and asteroidal sets.

3.1 Introduction

A dominating target is a set of vertices such that every connected superset is a dominating set. The dominating target number of G, denoted by $\mathrm{dt}(G)$, is the cardinality of a smallest dominating target. Special graphs with bounded $\mathrm{dt}(G)$ have been studied for their diametral and domination properties. For example, Kloks et al. [17] studied the spanning trees of such graphs and Aggarwal et al. [1] investigated classical algorithmic problems on graphs with small $\mathrm{dt}(G)$.

A significant connection has been established between dominating targets and *asteroidal sets* (see Definition 38). In particular, much is known about the class of asteroidal triple-free (AT-free) graphs, the class of graphs having $\mathrm{an}(G) \leq 2$. Notably, these graphs form a common superclass of the *interval* and *co-comparability* graphs. Corneil et al. [8] proved that AT-free graphs have a *dominating pair*, a dominating target of size at most 2. Since that discovery, the structure of graphs with dominating pairs and asteroidal sets has gained attention [2, 24], and in this paper we investigate the more general relationship between a dominating target and an asteroidal set of arbitrary size. The results of our paper are organized in the following way.

In Sect. 3.3, we show that asteroidal sets impose strong restrictions on dominating targets. For example, we prove that every connected superset of a dominating target contains a (quickly identifiable) vertex set of size at most $\mathrm{dt}(G)$ that dominates any given asteroidal set.

In Sect. 3.4, we investigate the notion of polarity of dominating targets. Whereas similar results have been proved for graphs having dominating pairs [8, 7, 24], no such results have been shown for graphs with $\mathrm{dt}(G) \geq 3$. We generalize the concept of a *polar pair* (see Definitions 34 and 49) to that of a *polar target*. Our main result is to prove that if a graph G contains a dominating triple that is also a 4-distant asteroidal triple then G has a polar target of size 3. This result provides algorithmic benefits and can be used to reduce the time-complexity of computing dominating triples.

3.2 Preliminaries

Our graph theory notation basically follows that of Golumbic [14] and of Chapter 2, with some adjustments (i.e. the definition of an asteroidal set accounts for sizes 1 and 2). Our graphs are always simple, undirected, and finite. When no risk of confusion arises, we will use V to denote the vertex set of G and E to denote its edge set. We let $n = |V|$. We write $u \sim v$ to indicate that vertices u and v are adjacent. More generally if $S, T \subseteq V$ we write $S \sim T$ if there exists $u \in S$ and $v \in T$ such that $u \sim v$.

As is standard, if $S \subseteq V$ we write $N(S) = \{y : \exists\, x \in S \text{ such that } x \sim y\}$ for the (open) neighborhood of S and $N[S] = N(S) \cup S$ for the closed neighborhood of S. A set S *dominates* a set $T \subseteq V$ if $T \subseteq N[S]$. We denote by $C_N(S)$ the *common neighborhood* of S. Formally,

$$C_N(S) = \bigcap_{v \in S} N(v).$$

A sequence of vertices $P = \langle u = x_0, x_1, \ldots, x_{k-1}, x_k = v \rangle$ is called a *walk* if $x_i \sim x_{i+1}$ for all $i \in \{0, 1, \ldots, k-1\}$. We say P is a path if the vertices $x_0, x_1, \ldots, x_k$ are all distinct. A walk with endpoints u and v is called a u, v-walk. If $v \sim v'$ then we denote by $P{-}v'$ the walk $\langle u = x_0, x_1, \ldots, x_{k-1}, x_k = v, v' \rangle$. Similarly, if $P = \langle x_0, x_1, \ldots, x_k \rangle$ and $P' = \langle y_0, y_1, \ldots, y_q \rangle$ are walks and $x_k \sim y_0$, then we let $P{-}P' = \langle x_0, \ldots, x_k, y_0, \ldots, y_q \rangle$. We write $P[x_i, x_j]$ where $(0 \leq i \leq j \leq k)$ for the subwalk $\langle x_i, \ldots, x_j \rangle$ of P. It is well known that every u, v-walk contains a u, v-path.

We say a walk P *meets* some set S if $P \cap S \neq \emptyset$ and that P *avoids* S if $P \cap S = \emptyset$. The notation $d_G(u, v)$ denotes *geodesic distance*, the number of edges along a shortest path, between the vertices u and v in the graph G. The *diameter* of G is the greatest geodesic distance between any two vertices in G. The *eccentricity* $\mathrm{ecc}(v)$ of a vertex v is the greatest geodesic distance between v and some other vertex in G.

Definition 38. *An* asteroidal set A *is an independent set of vertices such that, for every vertex* $v \in A$*, there exists a path between any two remaining vertices of* $A \setminus \{v\}$ *that avoids* $N[v]$*. We call such a path an* asteroidal path*. The cardinality of the largest asteroidal set in a graph is known as the* asteroidal number *of that graph, denoted by* $\mathrm{an}(G)$*. An* asteroidal triple *is an asteroidal set of size* 3.

Definition 39. *A set* $\{x_1, x_2, \ldots, x_k\}$ *is* k-distant *if* $d_G(x_i, x_j) \geq k$ *for all* $i, j \in [1, k]$ *such that* $i \neq j$.

Definition 40. *A* dominating target T *is a set of vertices such that every connected superset of* T *is a dominating set.*

The standard definition of a a *dominating pair* is that it is an ordered pair of vertices such that any path between them is dominating. In particular, a dominating pair need not correspond to a dominating target of size 2, it might have size 1. Similarly, in the literature, a *dominating triple* is a dominating target of size at most 3. More recently, dominating triples have been studied for their diameter properties [9]. It will be convenient for our proofs to have a name for a special combination of properties a triple might have:

Definition 41. *A triple is* golden *if it is a dominating triple and a* 4*-distant asteroidal triple.*

3.3 Asteroidal Sets and Dominating Targets

In this section, we explore the interaction between asteroidal sets and connected supersets of dominating targets.

Definition 42. *Let* $D = \{x_1, x_2, \ldots, x_k\}$ *be a dominating target, let* P *be an* x, y*-path for some* $x, y \in D$*, and let* A *be an asteroidal set. We denote by* f^P *(*resp. ℓ^P*) the first (*resp. *last) vertex of* P *that is adjacent to* A*. These exist when* P *meets any vertex neighboring* A*. We let* $F^P = N(f^P) \cap A$ *and* $L^P = N(\ell^P) \cap A$*.*

Remark. Note that any given path P in Definition 42 does not necessarily meet any vertex in the neighborhood of A. Then, neither f^P nor ℓ^P exist and we have that $F^P = L^P = \emptyset$.

The relationship between $an(G)$ and the size of some dominating target in G can be expressed by the following known result.

Proposition 43. [17] *Every graph* G *has* $\mathrm{dt}(G) \leq \mathrm{an}(G)$*.*

For the purpose of better understanding a dominating target D, we consider a minimal, general form of any connected superset of D. Such a superset satisfies a special property.

Lemma 44. *Given a dominating target* $D = \{x = x_1, x_2, \ldots, x_k\}$ *and an asteroidal set* A*, then for every* $\{P_2, P_3, \ldots, P_k\}$ *consisting of an* x, x_i*-path for each* $2 \leq i \leq k$*, we have*

$$\bigcup_{i \in [2,k]} F^{P_i} \cup L^{P_i} = A.$$

Proof. Suppose otherwise. There exists a $\{P_2, \ldots, P_k\}$ and $a \in A$ with a not in the above union. Among all such collections of paths with a not in the union, let $\{R_2, R_3, \ldots, R_k\}$ be one such that the smallest number of the paths dominate a. Since D is a dominating target, at least one of the paths dominates a.

Let R_i be a path that is adjacent to a. Since a is dominated by R_i but not in either F^{R_i} or L^{R_i} we must have that $f^{R_i} \neq \ell^{R_i}$ and that all vertices of R_i dominating a are strictly between f^{R_i} and ℓ^{R_i} on R_i. Let $b \in F^{R_i}$ and $c \in L^{R_i}$.

If $b \neq c$, then we let Q be an asteroidal b, c-path that does not dominate a, which exists because any three vertices in A are an asteroidal triple by Definition 40 and thus $\{a, b, c\}$ is an asteroidal triple. (If $b = c$, then we let Q be the one vertex path $\langle b \rangle$.) Let $W = R_i[x, f^{R_i}]{-}Q{-}R_i[\ell^{R_i}, x_i]$ and let R_i' be any x, x_i-path that can be extracted from the walk W. Then $\{R_2, R_3, \ldots, R_{i-1}, R_i', R_{i+1}, \ldots, R_k\}$ has fewer paths dominating a, a contradiction to our choice of $\{R_2, \ldots, R_k\}$. □

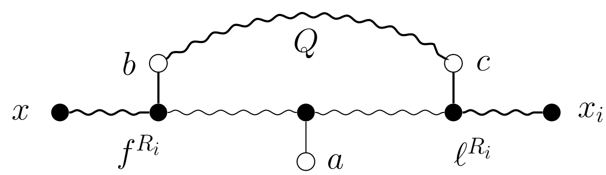

Figure 3.1: We depict the proof of Lemma 44. The bold path is R_i' and the unfilled vertices are an asteroidal triple.

Next, we strengthen the above lemma for all paths beginning with x and ending in $D \setminus \{x\}$. Together, these paths provide a stronger restriction on the connected supersets of D.

Lemma 45. *Given a dominating target $D = \{x = x_1, x_2, \ldots, x_k\}$ and an asteroidal set A, let $\mathscr{P}$ be the collection of all x, x_i-paths for any $2 \leq i \leq k$. Let*

$$\mathcal{F}_{D,x} = \bigcap_{\substack{P \in \mathscr{P} \\ F^P \neq \emptyset}} F^P, \qquad \mathcal{L}_{D,x} = \bigcup_{P \in \mathscr{P}} L^P.$$

Then some $P \in \mathscr{P}$ has $F^P \neq \emptyset$. Moreover, $\mathcal{F}_{D,x} \cup \mathcal{L}_{D,x} = A$.

Proof. We first show that some $P \in \mathscr{P}$ has $F^P \neq \emptyset$. Suppose not. Every $P \in \mathscr{P}$ has $F^P = \emptyset$. For all i, we take a single x, x_i-path P_i to form a collection of paths $\{P_2, P_3, \ldots, P_k\}$. By assumption, every $P \in \{P_2, \ldots, P_k\}$ has $F^P = \emptyset$. Thus, by

Lemma 44 we have that $L^{P_2} \cup L^{P_3} \cup \cdots \cup L^{P_k} = A$. So, some $L^{P_{i'}} \neq \emptyset$. Because $\ell^{P_{i'}}$ exists, certainly $f^{P_{i'}}$ exists. Therefore, $F^{P_{i'}} \neq \emptyset$, a contradiction.

Next, we will prove the second part of the lemma. Suppose for the sake of contradiction that there exists a vertex $a \in A \setminus (\mathcal{F}_{D,x} \cup \mathcal{L}_{D,x})$. In particular, there exists $a \in A$ such that $a \notin \mathcal{L}_{D,x}$, i.e. there exists no x, x_i-path P with $a \in L^P$. Since $a \notin \mathcal{F}_{D,x}$, there exists $2 \leq j \leq k$ and an x, x_j-path P_j such that $F^{P_j} \neq \emptyset$ and $a \notin F^{P_j}$. W.l.o.g. we may assume $j = k$. For all $2 \leq i \leq k-1$, pick an arbitrary x, x_i-path P_i.

By Lemma 44, we have that $\bigcup F^{P_i} \cup L^{P_i} = A$ for each $i \in [2, k]$. For each $i < k$, we will create a replacement path R_i. We will construct a set of paths $\{R_2, R_3, \ldots, R_{k-1}\}$ such that the union of the R_i's together with P_k is a connected superset of D that does not dominate a. This contradiction will prove the result.

For $2 \leq i \leq k-1$, if $L^{P_i} = \emptyset$ then let $R_i = P_i$. Otherwise, we know that F^{P_i} and L^{P_i} are non-empty. Pick $b \in F^{P_k}$ and $c_i \in L^{P_i}$, noting that neither b nor c_i is equal to a. Let Q_i be an asteroidal b, c_i-path that avoids $N[a]$. (If $b = c_i$ then let Q_i be the path $\langle b \rangle$). Now let $R_i = P_k[x, f^{P_k}]{-}Q_i{-}P_i[\ell^{P_i}, x_i]$. The union of the R_i's together with P_k forms a connected superset of D that does not dominate a, a contradiction. □

For examples of Lemma 45 where $\mathcal{F}_{D,x}$ is non-empty and empty, see Fig. 3.2 and Fig. 3.3, respectively. We immediately make the following observation.

Corollary 46. *Under the hypotheses of Lemma 45, if $\mathcal{F}_{D,x}$ is non-empty then either $D \cap C_N(\mathcal{F}_{D,x})$ is non-empty or $C_N(\mathcal{F}_{D,x})$ is a cut-set in G.*

Proof. Assume that $\mathcal{F}_{D,x}$ is non-empty and that $D \cap C_N(\mathcal{F}_{D,x})$ is empty. Let $y \in D \setminus \{x\}$. Every x, y-path meets $C_N(\mathcal{F}_{D,x})$. Thus $G \setminus C_N(\mathcal{F}_{D,x})$ contains the vertices x, y but they are in different components. □

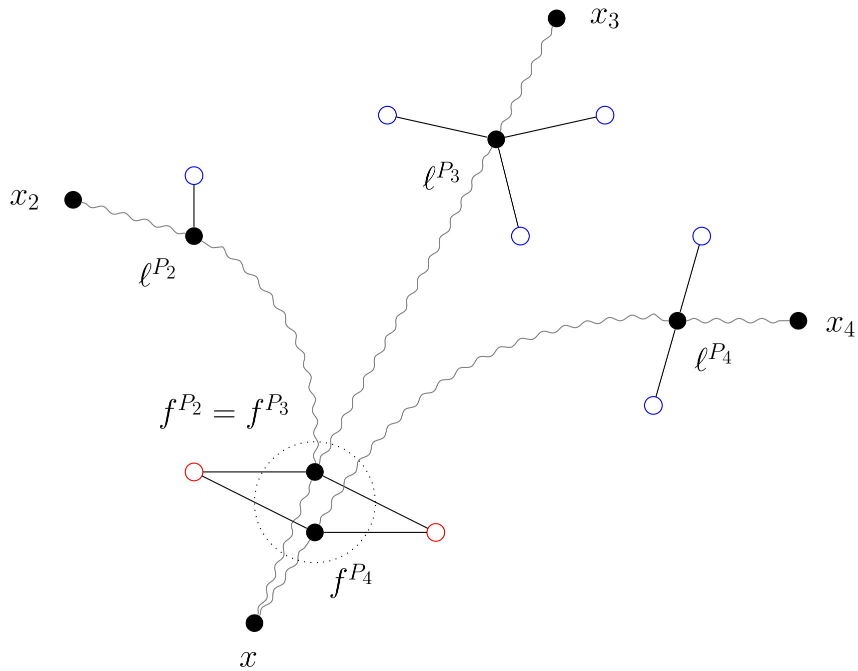

Figure 3.2: Let $D = \{x = x_1, \ldots, x_4\}$ be a dominating target and let the unfilled circles be an asteroidal set. Let P_2, P_3, P_4 be paths beginning at x and ending at x_2, x_3, x_4, respectively. The red vertices form set $\mathcal{F}_{D,x}$ and the blue vertices form set $\mathcal{L}_{D,x}$ (see Lemma 45). We have circled $C_N(\mathcal{F}_{D,x})$ for clarity.

Theorem 47 generalizes a result from Sect. 2.3 with $\mathrm{dt}(G) \leq 2$ (see Theorem 28).

Theorem 47. *Under the hypotheses of Lemma 45, we have that* $\mathcal{F}_{D,x_1} \cup \mathcal{F}_{D,x_2} \cup \cdots \cup \mathcal{F}_{D,x_k} = A$.

Proof. Suppose otherwise and that $a \in A \setminus (\mathcal{F}_{D,x_1} \cup \mathcal{F}_{D,x_2} \cup \cdots \cup \mathcal{F}_{D,x_k})$. Thus, for all $1 \leq i \leq k$ there exists a path P_i from x_i to $D \setminus \{x_i\}$ such that $F^{P_i} \neq \emptyset$ by Lemma 45 and $a \notin F^{P_i}$. Let $R_i = P_i[x_i, f^{P_i}]$ and let M be a connected superset of A formed from the asteroidal paths between pairs in A that avoid $N[a]$. Notice that each $f^{P_i} \in R_i$ is adjacent to M. The subgraph induced by the union of M and the R_i's forms a connected superset of D that does not dominate a, a contradiction. □

Remark Theorem 47 clearly implies that every connected superset of D contains

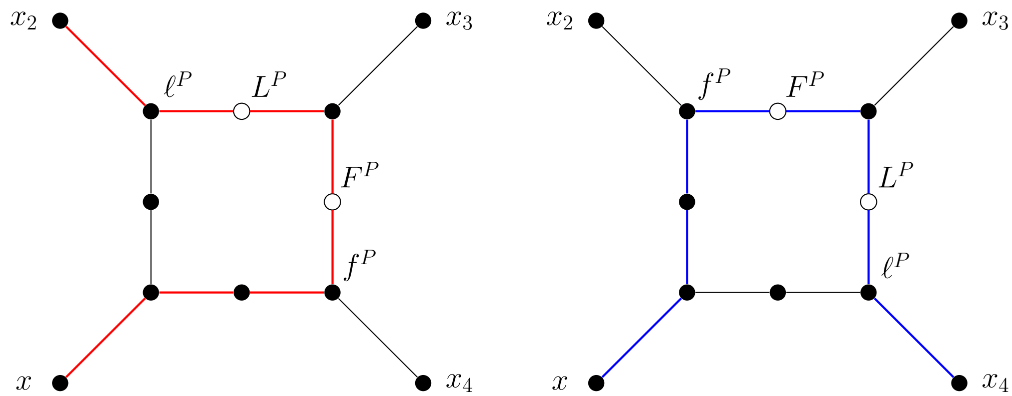

Figure 3.3: Let the unfilled vertices be an asteroidal set A and let $D = \{x = x_1, x_2, x_3, x_4\}$ be a dominating target. On the left (*resp.* right) we show an x, x_2-path (*resp.* x, x_4-path) P using red (*resp.* blue) edges. Although $\mathcal{F}_{D,x} = \emptyset$, we have that $A = \mathcal{L}_{D,x}$ as evidenced by the colored paths, so the second conclusion of Lemma 45 is satisfied. Notice that $A = \mathcal{F}_{D,x_3}$ satisfies Theorem 47. It is also satisfied by $A = \mathcal{F}_{D,x_2} \cup \mathcal{F}_{D,x_4}$.

a vertex set of size at most $\text{dt}(G)$ that dominates A.

Intuitively, we have shown that all connected supersets of a dominating target D are necessarily "funneled" through $C_N(\mathcal{F}_{D,x_i})$ for each i that has a non-empty $\mathcal{F}_{D,x_i}$. For instance, if a graph has $\text{dt}(G) < \text{an}(G)$, then there exists an i and a corresponding set $\mathcal{F}_{D,x_i}$ of size at least two. Then, $C_N(\mathcal{F}_{D,x_i})$ is more greatly restricted. Corollary 46 tells us that the removal of $C_N(\mathcal{F}_{D,x_i})$ either removes x_i or disconnects x_i from the rest of the graph.

3.4 Polar Targets

In this section, we generalize the concept of a polar pair to graphs with $\text{dt}(G) > 2$. We define the notion of a *polar target* (see Definition 49). We give a sufficient (but not necessary) condition for the existence of a polar target of size 3. The following definition is equivalent to that appearing in [8], where the notion of polar pairs was studied (note that the authors introduced the concept, but not the term polar pair).

Definition 48. *A pair (X, Y), where $X, Y \subseteq V$, is* polar *if $X \cap Y = \emptyset$ and a pair of vertices x, y is a dominating pair if and only if exactly one of them is in each of X and Y.*

It has been proven that AT-free graphs of diameter at least 4 have a polar pair [8]. It has also been shown that graphs having both a dominating pair and a diameter of at least 5 have a polar pair [24]. At any lower diameter, a polar pair is not guaranteed to exist. Certainly, a polar pair does not exist if $dt(G) \geq 3$. Therefore, we introduce the concept of polar targets as an analog to polar pairs for graphs with $\mathrm{dt}(G) > 2$.

Definition 49. *Given a graph with $dt(G) = k \geq 2$, a disjoint collection $X_1, X_2, \ldots, X_k$ of subsets of V is a* polar target *if the set of all dominating targets of size $\mathrm{dt}(G)$ is precisely the collection of all sets D having $|D \cap X_i| = 1$ for all i.*

Chordless cycles are interesting as they have no polar target of any size.

Proposition 50. *For all $n \geq 3$, we have $dt(C_n) = \lceil n/3 \rceil$.*

Proof. Clearly there is a set of vertices $S = \{v_1, v_2, \ldots, v_k\}$ where $k = \lceil n/3 \rceil$, $d(v_i, v_{i+1}) = 3$ for $i = 1, 2, \ldots, k-1$ and $d(v_k, v_1) \geq 3$. S is a dominating target, so $\mathrm{dt}(C_n) \leq \lceil n/3 \rceil$. Let $T \subseteq V(C_n)$ have size $t < n/3$. Label the vertices of T cyclically as $\{w_1, w_2, \ldots, w_t\}$. Then

$$3t < n = \sum_{i=1}^{t} d(w_i, w_{i+1}),$$

where indices are computed cyclically. Hence at least one of these distances is at least 4. If $d(w_i, w_{i+1}) \geq 4$ then a path from w_{i+1} to w_i going the long way around the cycle is a connected superset of T but not a dominating set. Thus $\mathrm{dt}(C_n) \geq \lceil n/3 \rceil$. □

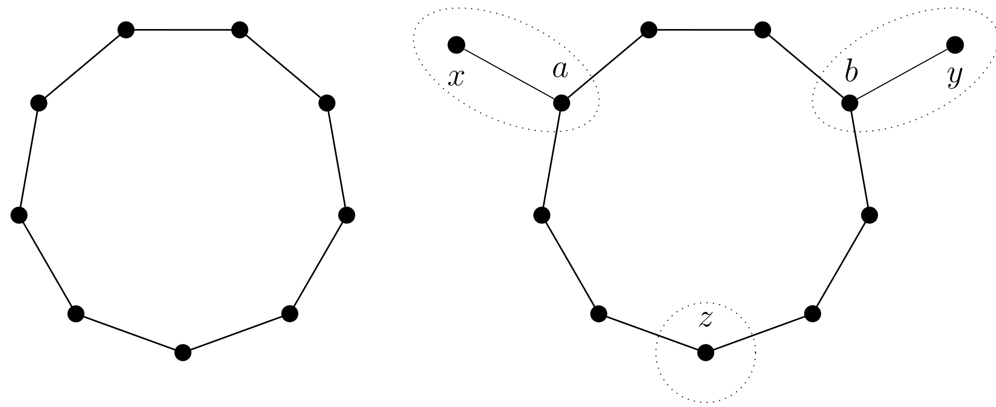

Figure 3.4: The left graph is a cycle on 9 vertices that does not contain a polar target of any size. Notice that every dominating triple is 3-distant, and no larger cycle graph exists such that $\mathrm{dt}(G) = 3$ by Proposition 50. The right graph contains a dominating triple (x, y, z) that is golden. A polar target $\{X_1, X_2, X_3\}$ of size $\mathrm{dt}(G) = 3$ exists where $X_1 = \{x, a\}$, $X_2 = \{y, b\}$, and $X_3 = \{z\}$.

Example 50.1. *For all $k \geq 2$ if we let $n = 3k$ then C_n has $\mathrm{dt}(C_n) = k$ but no polar target. In particular, if D is a dominating target of size k then so is D' obtained by shifting every vertex of D clockwise by 1 (see the left graph in Fig. 3.4).*

The main result of this section, Theorem 57, gives a sufficient condition for the existence of a polar target of size 3 in graphs with $\mathrm{dt}(G) = 3$. It builds on a sequence of lemmas which we will now begin.

Lemma 51. *Let G be a graph with $\mathrm{dt}(G) = 3$ and let (x_1, x_2, x_3), (y_1, y_2, y_3) be dominating triples such that (x_1, x_2, x_3) is golden. Let S be a connected superset of (y_1, x_2, x_3) that does not dominate d. Then there exists $i \in \{2, 3\}$ such that all x_2, y_i-paths and all x_3, y_i-paths dominate d.*

Proof. Suppose otherwise. W.l.o.g. let P be an induced x_2, y_2-path that does not dominate d. Either there exists an induced x_2, y_3-path that does not dominate d or there exists an induced x_3, y_3-path that does not dominate d. Let P' be such a path in either case. Then, $S \cup P \cup P'$ induces a connected superset of (y_1, y_2, y_3) that does not dominate d, a contradiction. $\square$

Next, we consider the interaction between a golden triple and any other given dominating triple.

To assist in our proof, for an asteroidal triple $\{a, b, c\}$ let $A^c_{a,b}$ denote an asteroidal a, b-path that avoids $N[c]$. More generally, let $a \leftrightarrow^c b$ denote an a, b-walk that avoids $N[c]$. Finally, we will borrow the terminology of a *d-octopus*, originally given by Fomin et. al. [12], in order to describe connected supersets of a dominating target. We consider a weaker variant, which we define as follows:

Definition 52. *A* weak d-octopus *of a graph is a dominating subgraph whose vertices are the union of d walks that have one endpoint in common.*

We are prepared to begin the lemma.

Lemma 53. *Given a graph G with $\mathrm{dt}(G) = 3$, let $T = (x, y, z), T' = (a, b, c)$ be dominating triples such that T is golden. Either (a, y, z), (b, y, z), or (c, y, z) is a dominating triple.*

Proof. Suppose otherwise. Let $P_{a,y}, P_{a,z}$ be induced paths with endpoints (a, y), (a, z), respectively, whose union is a connected superset of (a, y, z) that does not dominate some vertex u. Similarly, there exist induced paths $P_{b,y}, P_{b,z}$ and $P_{c,y}, P_{c,z}$ with endpoints $(b, y), (b, z)$ and $(c, y), (c, z)$ whose union does not dominate v and w, respectively.

The union of these six named paths is a dominating, connected superset of T' so at least one of the these paths must be adjacent to x. W.l.o.g. suppose that $x \sim P_{a,z}$. Now $P_{a,y} \cup P_{a,z} \cup \{x\}$ is a dominating, connected superset of T. Therefore, $x \sim u$ because we have assumed that $x \not\sim (P_{a,y} \cup P_{a,z})$. By Lemma 51, where $(x_1, x_2, x_3) := T$,

$(y_1, y_2, y_3) := T'$, $S := (P_{a,y} \cup P_{a,z})$, and $d := u$, there exists an $i \in \{b, c\}$ such that all y, i-paths and all z, i-paths dominate u. W.l.o.g. suppose that all y, b-paths and all z, b-paths dominate u.

Let f_x be the first vertex in $P_{a,z}$ that is adjacent to x and let f_u be the first vertex in $P_{b,y}$ that is adjacent to u. By assumption, any path taking the form of $\langle x, u, f_u, \ldots, y\rangle$ has length 4 or more and thus $d_G(f_u, y) > 1$. At this stage of the proof, our setup matches the left graph shown in Fig. 3.5. Let $a \leftrightarrow^z x = P_{a,z}[a, f_x]$ $-x$, $x \leftrightarrow^y b = \langle x, u\rangle - P_{b,y}[f_u, b]$, and $R = A^y_{x,z} - P_{c,z}$. Thus, $a \leftrightarrow^z x, x \leftrightarrow^y b, R$ (each walk having an endpoint x) establish a weak 3-octopus containing T' where y may only be adjacent to vertices in $P_{a,z}[a, f_x]$ or $P_{c,z}$.

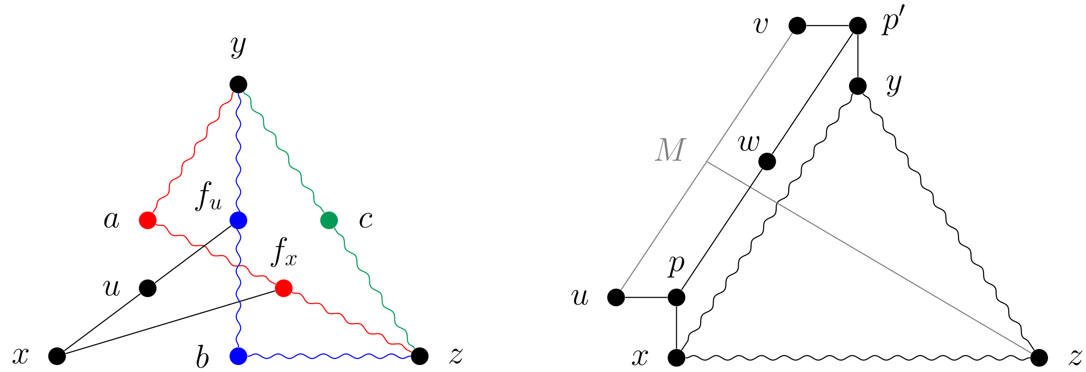

Figure 3.5: The left graph shows the setup of Lemma 53 right before entering Cases 1 or 2. We depict the initial six named paths $P_{a,y}, \ldots, P_{c,z}$ with a as an endpoint in red, b as an endpoint in blue, and c as an endpoint in green. The right graph depicts the proof of Lemma 55.

Case 1: $y \sim P_{c,z}$ and $y \not\sim P_{a,z}[a, f_x]$. Let f_y be the first vertex in $P_{c,z}$ that is adjacent to y and let f'_u be the first vertex in $P_{b,z}$ that is adjacent to u. Notice that $f_y \not\sim z$, otherwise $d_G(y, z) = 2$. Next, let $x \leftrightarrow^z b = \langle x, u\rangle - P_{b,z}[f'_u, b]$ and $x \leftrightarrow^z c = A^z_{x,y} - P_{c,z}[f_y, c]$. Now $a \leftrightarrow^z x, x \leftrightarrow^z b, x \leftrightarrow^z c$ (each walk having an endpoint x) establish a weak 3-octopus containing T' that does not dominate z, a contradiction.

Case 2: $y \sim P_{a,z}[a, f_x]$. Let f_y be the first vertex in $P_{a,z}$ that is adjacent to y. In particular, note that $f_y \neq f_x$, otherwise $d_G(x, y) = 2$. Next, let $a \leftrightarrow^x z = P_{a,z}[a, f_y] - A^x_{y,z}$. Now $a \leftrightarrow^x z, P_{b,z}, P_{c,z}$ (each walk having a endpoint z) establish a weak 3-octopus containing T' such that it is only possible that x is adjacent to $P_{b,z}$ or $P_{c,z}$.

Case 2.1: $x \sim P_{c,z}$ and $x \not\sim P_{b,z}$. Let f'_x be the first vertex in $P_{c,z}$ that is adjacent to x. Notice that $f'_x \not\sim z$, otherwise $d_G(x, z) = 2$. Let f'_u be the first vertex in $P_{b,z}$ that is adjacent to u (noting that $f'_u \not\sim z$), let $c \leftrightarrow^z x = P_{c,z}[c, f'_x] - x$, and let $x \leftrightarrow^z b = \langle x, u\rangle - P_{b,z}[f'_u, b]$. Now $a \leftrightarrow^z x, c \leftrightarrow^z x, x \leftrightarrow^z b$ (each walk having endpoint x) establish a weak 3-octopus containing T' that does not dominate z, a contradiction.

Case 2.2: $x \sim P_{b,z}$. We have that $P_{b,y} \cup P_{b,z} \cup \{x\}$ is a connected superset of T. We have that $v \sim x$ because $v \not\sim (P_{b,y} \cup P_{b,z})$ by assumption. By Lemma 51, where $(x_1, x_2, x_3) := T$, $(y_1, y_2, y_3) := (b, a, c)$, $S := (P_{b,y} \cup P_{b,z})$, and $d := v$, we have that there exists an $i' \in \{a, c\}$ such that all y, i'-paths and all z, i'-paths dominate v.

We will show that $i' = a$. To do so, we suppose that some induced z, c-path dominates v. Let f_v be the first vertex in $P_{c,z}$ that is adjacent to v. Note that $f_v \not\sim z$, otherwise $d_G(x, z) \leq 3$ by the existence of path $\langle x, v, f_v, z\rangle$. There exist $x \leftrightarrow^z c = \langle x, v\rangle - P_{c,z}[f_v, c]$ and $x \leftrightarrow^z b = \langle x, u\rangle - P_{b,z}[f'_u, b]$. Now $a \leftrightarrow^z x, x \leftrightarrow^z c, x \leftrightarrow^z b$ (each walk having endpoint x) establish a weak 3-octopus containing T' that does not dominate z, a contradiction.

Since we have previously assumed that $x \sim P_{b,z}$, let ℓ_x be the last vertex in $P_{b,z}$ that is adjacent to x. Consider the following walk: let $Q = P_{b,z}[b, \ell_x] - x$ and $Q' = x - A^z_{x,y} - P_{c,y}$. The walks $a \leftrightarrow^z x, Q, Q'$ (each walk having endpoint x) establish a weak 3-octopus containing T' such that z can only be dominated by the subpath

$P_{c,y}$. We let ℓ_z be the last vertex in $P_{c,y}$ that is adjacent to z. Finally, we consider three different cases where $P_{a,z}$ is adjacent to v. We let ℓ_y be the last vertex in $P_{a,z}$ that is adjacent to y.

Case 2.2.1: Some vertex after ℓ_y along $P_{a,z}$ is adjacent to v. Let ℓ_v be the last such vertex in $P_{a,z}$ (recall that $i' = a$). Also, let ℓ'_v be the last vertex in $P_{a,y}$ that is adjacent to v and let ℓ_u be the last vertex in $P_{b,y}$ that is adjacent to u. Notice that $\ell'_v \not\sim y$, otherwise $\langle y, \ell'_v, v, x\rangle$ is an, x, y-path of length 3, a contradiction. Similarly, $\ell_u \not\sim y$. Let $a \leftrightarrow^y v = P_{a,y}[a, \ell'_v]{-}v$, $v \leftrightarrow^y z = v{-}P_{a,z}[\ell_v, z]{-}P_{c,y}[\ell_z, c]$, and $v \leftrightarrow^y b = \langle v, x, u\rangle{-}P_{b,y}[\ell_u, b]$. Now $a \leftrightarrow^y v, v \leftrightarrow^y z, v \leftrightarrow^y b$ (each walk having endpoint v) establish a weak 3-octopus containing T' that does not dominate y, a contradiction.

Case 2.2.2: Some vertex before f_y along $P_{a,z}$ is adjacent to v. Let f'_v be the first such vertex in $P_{a,z}$. Let ℓ_u be the last vertex in $P_{b,y}$ that is adjacent to u. Let $a \leftrightarrow^y x = P_{a,z}[a, f'_v]{-}\langle v, x\rangle$, $x \leftrightarrow^y b = \langle x, u\rangle{-}P_{b,y}[\ell_u, b]$, and $x \leftrightarrow^y c = x{-}A^y_{x,z}{-}P_{c,y}[\ell_z, c]$. Now $a \leftrightarrow^y x, x \leftrightarrow^y b, x \leftrightarrow^y c$ (each walk having endpoint x) establish a weak 3-octopus containing T' that does not dominate y, a contradiction.

Case 2.2.3: Every vertex along $P_{a,z}$ that is adjacent to v is contained in $P_{a,z}[f_y, \ell_y]$. Let $a \leftrightarrow^z y = P_{a,z}[a, f_y]{-}y{-}P_{a,z}[\ell_y, z]$. Now $a \leftrightarrow^z y, P_{b,z}, P_{c,z}$ (each walk having endpoint z) establish a weak 3-octopus containing T' that does not dominate v, a contradiction. □

Definition 54. *Given a graph G with $\mathrm{dt}(G) = 3$, let $T = (x, y, z)$ be a golden triple and let $T' = (a, b, c)$ be a dominating triple. Then we say that $a \in T'$ is* associated with *$x \in T$ if (a, y, z) is a dominating triple.*

In the next lemma, we show that in fact under the hypotheses of Lemma 53 there is a bijection between T and T' such that each element of T' is associated with its image in T. In fact we prove this by showing that we can make a double substitution from T' into T.

Lemma 55. *Under the hypotheses of Lemma 53, suppose that* (u, y, z) *and* (x, v, z) *are dominating triples with* $u, v \in \{a, b, c\}$*. Then,* $\{u, v, z\}$ *is a dominating target, and in particular* $u \neq v$*. From this it is immediate that there is a bijection mapping* $\{x, y, z\}$ *to* $\{a, b, c\}$ *such that each of* x, y, z *is associated with its image.*

Proof. Suppose otherwise. Let M be a connected superset of (u, v, z) that is not adjacent to some vertex w. Note that we will be especially concerned with the possibility that $w = x$ or $w = y$. Let $A^z_{x,y}, A^y_{x,z}$, and $A^x_{y,z}$ be induced asteroidal paths. The walks $W = A^z_{x,y}{-}A^x_{y,z}$ and $W' = A^y_{x,z}{-}A^x_{y,z}$ are connected supersets of T and hence dominate G.

Let $p \in W$ be adjacent to u. Suppose $p \notin N[x]$. If $p \in A^z_{x,y}$ then $u{-}A^z_{x,y}[p, y]$ ${-}A^x_{y,z}$ is a connected superset of (u, y, z) that does not dominate x. Otherwise, if $p \in A^y_{x,z}$ then $u{-}A^y_{x,z}[p, z]{-}A^x_{y,z}$ is a connected superset of (u, y, z) that does not dominate x. Either case gives us a contradiction. Therefore, $p \in N[x]$. Similarly, for all $p' \in W'$ adjacent to v we have $p' \in N[y]$.

Now $M \cup \langle v, p', y\rangle$ is a connected superset of (u, y, z) such that only $\langle p', y\rangle$ can be adjacent to w. Therefore, $w \in N[\{p', y\}]$. (Note that this is even true if $y = p' = w$). Similarly, $w \in N[\{p, x\}]$. There is, therefore, a path of length at most 4 from x to y going via w. If it is not the induced path $P = \langle x, p, w, p', y\rangle$ (with all these vertices distinct), then $d_G(x, y) < 4$, a contradiction. The right graph in Fig. 3.5 depicts this stage of our proof.

The preceding establishes that $M \sim x$ and $M \sim y$. Consequently, $G[\{x\} \cup M \cup \{y\}]$ is a connected superset of T that does not dominate w, a contradiction. So far, we have proven (u, v, z) is a dominating triple. If $u = v$, then $(u = v, z)$ is a dominating pair, contradicting $\mathrm{dt}(G) = 3$. Therefore, $u \neq v$. Similarly, $z \notin \{u, v\}$. Thus, $\{u, v, z\}$ is a dominating target. □

We have shown that, given a golden triple T and an arbitrary dominating triple U, we can exchange vertices between T and U up to twice. Next we show by Theorem 56 that any other dominating triple U' can exchange vertices with either T or U. Two lemmas that are applied within the proof can be found in the Appendix.

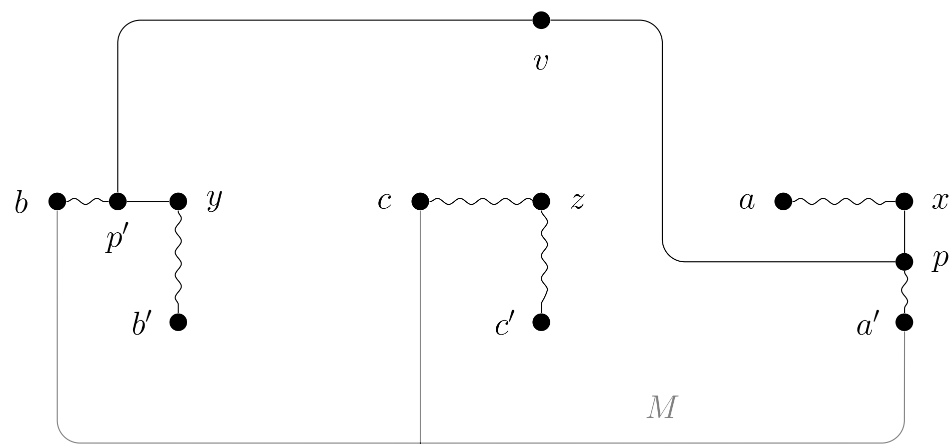

Figure 3.6: Portraying the proof of Theorem 56 where $w = c$ and $v \sim P_{b,y}$.

Theorem 56. *Given a graph G with $\mathrm{dt}(G) = 3$, let $T = (x, y, z)$, $U = (a, b, c)$, and $U' = (a', b', c')$ be dominating triples such that T is golden. Suppose that a, a' are associated with x, b, b' are associated with y, and c, c' are associated with z. Then any set W containing exactly one element from each of $\{x, a, a'\}$, $\{y, b, b'\}$, $\{z, c, c'\}$ is a dominating triple.*

Proof. If $|\{x, y, z\} \cap W| \geq 1$ then we are done by Lemmas 53 and 55. Without loss of generality it suffices to prove that (a', b, w) for each $w \in \{z, c, c'\}$ is a dominating triple.

Suppose for the sake of contradiction that some (a', b, w) is not a dominating triple. Some connected superset M of (a', b, w) does not dominate some vertex v. Let

$P_{a',x}, P_{b,y}, P_{w,z}$ be induced paths with endpoints $(a', x), (b, y), (w, z)$, respectively. By Lemma 58, we have that $v \notin T$. If $w \in \{z, c\}$ then, by assumption, $M \cup P_{a',x}$ is a connected superset of (x, b, w) and is hence dominating. In particular, $v \sim P_{a',x}$. If instead $w = c'$, then similarly $M \cup P_{b,y}$ is a connected superset of the dominating triple (a', y, c') and thus $v \sim P_{b,y}$. These two cases are symmetric (switching $\{x, a, a'\}$ for $\{y, b, b'\}$) so w.l.o.g. we assume $w \in \{z, c\}$.

Let ℓ_v be the last vertex in $P_{a',x}$ that is adjacent to v. Then $P_{b,y} \cup M \cup P_{w,z}$ is a connected superset of the dominating triple (a', y, z), so $v \sim P_{b,y}$ or $v \sim P_{w,z}$. If $w = z$ then $v \not\sim P_{w,z}$ because $v \in N[z]$ implies that $v \in N[M]$, a contradiction. Therefore $v \sim P_{b,y}$. If $w = c$, we could have either $v \sim P_{b,y}$ or $v \sim P_{w,z}$. These cases are symmetric, so we assume w.l.o.g. in both cases, $w = c$ or $w = z$, that $v \sim P_{b,y}$. Let ℓ'_v be the last vertex in $P_{b,y}$ that is adjacent to v.

By Lemma 59 where $(x_1, x_2, x_3) := (x, y, z)$, $(y_1, y_2, y_3) = (a', b, w)$, $M := M$, and $v := v$, let $p \in N[x] \cap N[v]$ satisfy the conclusion of the lemma.

By Lemma 59 where $(x_1, x_2, x_3) := (y, x, z)$, $(y_1, y_2, y_3) = (b, a', w)$, $M := M$, and $v := v$, let $p' \in N[y] \cap N[v]$ satisfy the conclusion of the lemma.

There is a path of length at most 4 from x to y going via v. If it is not the induced path $P = \langle x, p, v, p', y \rangle$ (with all these vertices distinct as shown in Fig. 3.6), then $d_G(x, y) < 4$, a contradiction. The preceding establishes that $M \sim x$ and $M \sim y$. Consequently, $G[\{x\} \cup M \cup \{y\}]$ is a connected superset of (x, y, w) that does not dominate v, a contradiction. □

The following immediate consequence of Theorem 56 is our main result.

Theorem 57. *Let G be a graph with* $\mathrm{dt}(G) = 3$. *If G contains a golden triple, then G has a polar target of size* 3.

Proof. Let $\{x, y, z\}$ be a golden triple and set $A = \{a \in V : a \text{ is associated with } x\}$ and similarly let B and C be the sets of vertices associated to y and z, respectively. By Theorem 56 any set meeting each of A, B, C in one vertex is a dominating triple. By Lemma 55, any dominating triple has this form. □

At is trivial to show that all dominating triples in a graph can be computed in $O(n^4)$. However, if a golden triple T is given, then by Theorem 57 we have that a polar target of size 3 exists. Then, all remaining dominating triples can be computed in $O(n^2)$ as follows: Any single set in the polar target can be computed by fixing a pair of vertices $x, y \in T$, iterating on every other vertex z in the graph, and checking if $\{x, y, z\}$ is a dominating triple. Thus, on the algorithmic side, a polar target provides reduced run-time complexities.

We believe that this result can be generalized to polar targets of arbitrary size.

Conjecture 57.1. *If $k \geq 2$ and G is a graph having $\mathrm{dt}(G) = k$ and a k-set that is simultaneously a 4-distant asteroidal set and a dominating target, then G has a polar target of size k.*

3.5 Appendix

This appendix contains two lemmas that are applied within Theorem 56 to complete the result. Lemma 58 can be precisely applied within Theorem 56 using the same names for variables. So, we keep the names in the lemma identical to the names given by the hypotheses of the theorem. The same cannot be done with Lemma 59, so the vertices within its hypotheses are given a more generic naming convention (i.e. $x_1, x_2, \ldots$).

Lemma 58. *Given a graph G with* $\mathrm{dt}(G) = 3$*, let* $T = (x, y, z)$*,* $U = (a, b, c)$*, and* $U' = (a', b', c')$ *be dominating triples such that T is golden. Suppose that a, a' are associated with x, b, b' are associated with y, and c, c' are associated with z. Let $w \in \{z, c, c'\}$ and suppose some connected superset M of (a', b, w) does not dominate a vertex v. Then $v \notin T$. (Of course, the same holds for $\{x, a, a'\}$, $\{y, b, b'\}$.)*

Proof. By Lemma 55 we have that any double substitution of associated vertices from U or from U' into T is a dominating triple.

Let $P_{a',x}, P_{b,y}, P_{w,z}$ be induced paths with endpoints $(a', x), (b, y), (w, z)$, respectively. Suppose for the sake of contradiction that $v \in T$.

Case 1. $v = x$ or $v = y$. W.l.o.g. let $v = y$. Suppose that $w \in \{z, c\}$. We have assumed that (x, b, w) is a dominating triple. Hence, $M \cup P_{a',x}$ must dominate y. Let ℓ_y be the last vertex in $P_{a',x}$ that is adjacent to y. Notice that $\ell_y \not\sim x$, otherwise $d_G(x, y) \leq 2$. However, the walk $W = P_{a',x}[a', \ell_y]{-}y{-}A^x_{y,z}$ is a connected superset of (a', y, z) that does not dominate x, a contradiction.

Instead, suppose that $w = c'$. We have that $M \cup P_{a',x} \cup P_{c',z}$ is a connected superset of (x, b, z) and, thus, dominates y. W.l.o.g. suppose that ℓ_y is the last vertex in $P_{a',x}$ that is adjacent to y. Similar to where $w \in \{z, c\}$, the walk W exists and it does not dominate x, a contradiction.

Case 2. $v = z$. Obviously, $w \neq z$. W.l.o.g. let $w = c$. Now $M \cup P_{a',x}$ is a connected superset of (x, b, c). Let ℓ_z be the last vertex in $P_{a',x}$ that is adjacent to z. Notice that $\ell_z \not\sim x$, otherwise $d_G(x, z) \leq 2$. However, the walk $W = P_{a',x}[a', \ell_z]{-}z{-}A^x_{z,y}$ is a connected superset of (a', y, z) that does not dominate x, a contradiction. □

Lemma 59. *Given a graph G with* $\mathrm{dt}(G) = 3$*, let* $T = (x_1, x_2, x_3)$*,* $U = (y_1, x_2, x_3)$*,* $U' = (x_1, y_2, y_3)$ *be dominating triples such that T is golden. Suppose M is some connected superset of (y_1, y_2, y_3) that does not dominate v. Let P_{y_1,x_1} be an induced*

y_1, x_1-path and that ℓ_v is the last vertex on it adjacent to v. Similarly, let ℓ'_v be the last vertex in an induced y_2, x_2-path P_{y_2,x_2} adjacent to v. Then $N[x_1] \cap N[v] \neq \emptyset$.

Proof. Let $R_1 = P_{y_1,x_1}[y_1, \ell_v]$ and $R_2 = P_{y_2,x_2}[\ell'_v, x_2]$. Then, let $W = R_1{-}v{-}R_2{-}A^{x_1}_{x_2,x_3}$. Now W is a connected superset of U such that only $\langle \ell_v, v\rangle$ or R_2 may neighbor x_1. If $x_1 \sim \ell_v$ or $x_1 \sim v$ then we are done.

Suppose that $x_1 \not\sim \ell_v, v$. Then, $x_1 \sim R_2$. Let ℓ_{x_1} be the last vertex in R_2 that is adjacent to x_1. If $\ell_{x_1} = \ell'_v$, then we are done. Otherwise, the walk $x_1{-}R_2[\ell_{x_1}, x_2]{-}A^{x_1}_{x_2,x_3}$ is a connected superset of T and thus must be adjacent to y_1.

Suppose that $x_1 \in N[y_1]$. Now $M \cup \langle y_1, x_1\rangle$ is a connected superset of U' and thus $x_1 \sim v$. There exists a walk $\langle v, x_1, y_1\rangle$, so we are done. Suppose that $x_1 \notin N[y_1]$. Any $A^{x_1}_{x_2,x_3}$ cannot be adjacent to y_1, otherwise $\{y_1\} \cup A^{x_1}_{x_2,x_3}$ is a connected superset of U that does not dominate x_1, a contradiction. Therefore, $y_1 \sim R_2[\ell_{x_1}, x_2]$.

Let ℓ_{y_1} be the last vertex in $R_2[\ell_{x_1}, x_2]$ that is adjacent to y_1. If $\ell_{x_1} \neq \ell_{y_1}$ then $y_1{-}R_2[\ell_{y_1}, x_2]{-}A^{x_1}_{x_2,x_3}$ is a connected superset of U that does not dominate x_1, a contradiction. Thus, $\ell_{x_1} = \ell_{y_1}$. Let $Q = \langle y_1, \ell_{y_1}, x_1\rangle$. Since $M \cup Q$ is a connected superset of U', we have that $v \sim \ell_{y_1}$ or $v \sim x_1$, so we are done. □

Chapter 4

Conclusion

In this thesis, we studied properties of graphs with asteroidal sets and dominating targets of both bound and arbitrary sizes. We presented properties of graphs that do not have an asteroidal triple with 3-spread, and showed that for such graphs every dominating pair path meets the common neighborhood of some pair in each asteroidal triple. Our notable results include a faster algorithm for the recognition of chordal hereditary dominating pair graphs, an improvement over the previous best run-time posed by Pržulj, Corneil, and Köhler [24]. Moreover, we corrected a mistake in the literature, showing by counterexample that a graph having a dominating shortest path does not necessarily contain a dominating diametral path. We defined strict dominating pair graphs in order to capture the discrepancy between these two properties, and we studied algorithmic and structural properties of strict dominating pair graphs. We showed that asteroidal quadruples are a key structural property in these graphs and that, even in general, asteroidal sets place strong restrictions on the connected supersets of dominating targets. These results accompany literature, as early as Kloks et al. [17], that has shown strong connections between these two types of sets.

Some of our results generalize known properties of AT-free graphs or graphs where dominating target number is at most 2. Notably, we defined the concept of a polar target, showed a sufficient condition for a graph to have a polar target of size 3, and conjectured that a similar condition may exist for a graph to have a polar target of size k. Along the way, we studied graphs that cannot have polar targets, most notably cycle graphs. Additional study of vertex eccentricities may lead to a necessary condition for polar targets to exist in graphs. The study of the diameter problem on strict dominating pair graphs could extend our result of Corollary 17 to a more general case. A possibility for the polynomial-time recognition of the HDP graphs remains an interesting open problem. Conditions for existence or non-existence of polar targets may provide additional, fruitful inquiry of dominating targets and their diameter properties.

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